Contextuality as a resource

simulations, adaptivity comonad, and the (partial) algebraic-logical view

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

Resources in Computation Workshop
University College London
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This talk

Based on joint work with Samson Abramsky, Martti Karvonen, Shane Mansfield

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- Sheaf-theoretic formalism for contextuality

  '*The sheaf-theoretic structure of non-locality and contextuality*'
  Abramsky & Brandenburger, New Journal of Physics, 2011.

  '*Logical Bell inequalities*'
  Abramsky & Hardy, Physical Review A, 2012.

  '*Contextuality, cohomology, and paradox*'
  Abramsky, B, Kishida, Lal, & Mansfield, CSL 2015.

  (cf. Cabello–Severini–Winter, Acín–Fritz–Leverrier–Sainz)

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- Resource theory for contextuality

  ‘*Contextual fraction as a measure of contextuality*’
  Abramsky, B, Mansfield, Physical Review Letters, 2017.

  ‘*Categories of empirical models*’
  Karvonen, QPL 2018.

  ‘*A comonadic view of simulation and quantum resources*’
  Abramsky, B, Karvonen, Mansfield, LiCS 2019.

  ‘*Closing Bell: boxing black box simulations in the resource theory of contextuality*’
  B, Karvonen, Mansfield, in Abramsky on Logic and Structure in CS and Beyond, Springer, 2022.

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Based on joint work with Samson Abramsky, Martti Karvonen, Shane Mansfield

- Partial Boolean algebras

  ‘*The logic of contextuality*’
  Abramsky & B, CSL 2021.

  ‘*Duality for transitive partial CABAs*’
  Abramsky & B, TACL 2022.

- Resource theory via pBAs

  ongoing work with Martti Karvonen

Overview

- Central object of study of quantum information and computation theory: the **advantage** afforded by **quantum resources** in information-processing tasks.

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- A hypothesis: this is related to **non-classical** features of quantum mechancics.
- **Contextuality** is a quintessential marker of non-classicality, an empirical phenomenon distinguishing QM from classical physical theories.

The essence of contextuality

- Not all properties may be observed at once.
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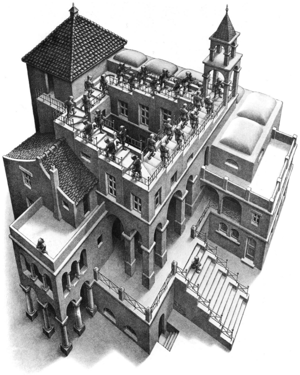

M. C. Escher, *Ascending and Descending*

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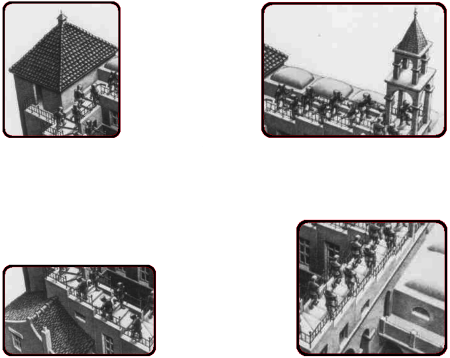

Local consistency

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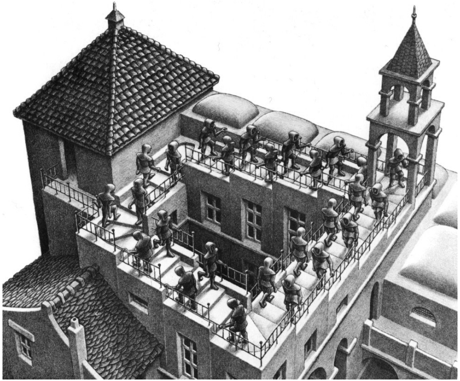

Local consistency *but* **Global inconsistency**

Contextuality and advantage in quantum computation

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- Measurement-based quantum computation (MBQC)

  ‘*Contextuality in measurement-based quantum computation*’
  Raussendorf, Physical Review A, 2013.

- Magic state distillation

  ‘*Contextuality supplies the ‘magic’ for quantum computation*’
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  ‘*Quantum advantage with shallow circuits*’
  Bravyi, Gossett, Koenig, Science, 2018.

  - Contextuality analysis: Aasnæss, Forthcoming, 2020.

Resource theory of contextuality

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- The 'free' operations are given by **classical procedures** $S \longrightarrow T$.

- We first consider non-adaptive procedures,
- and then capture **adaptivity** via a **comonadic** construction.

Contextuality

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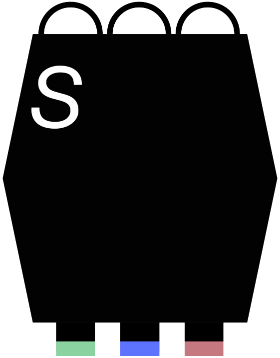

- Interaction with system: perform measurements and observe respective outcomes

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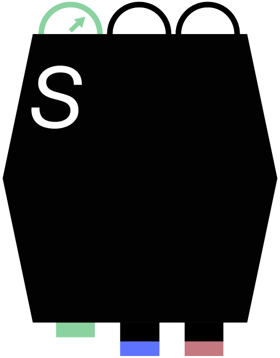

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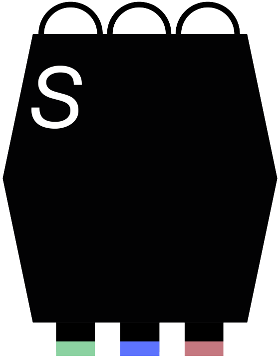

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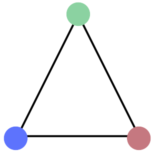

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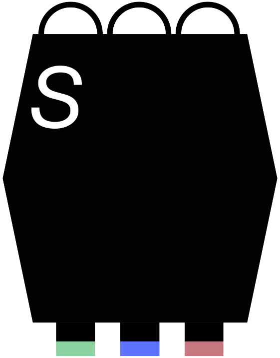

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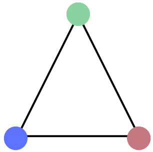

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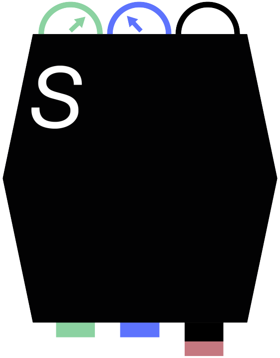

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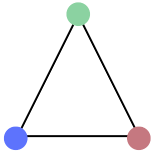

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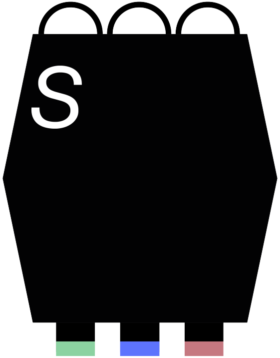

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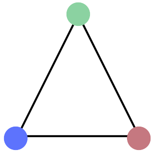

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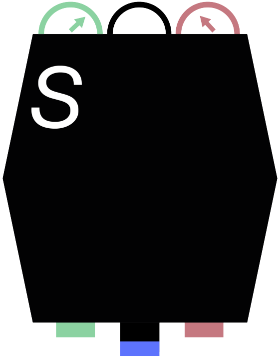

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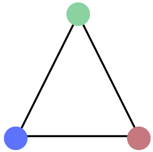

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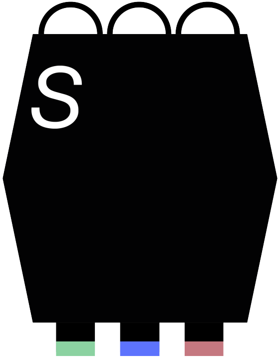

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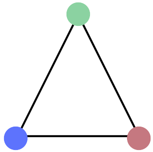

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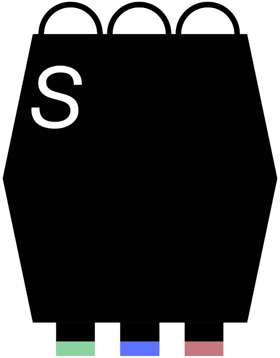

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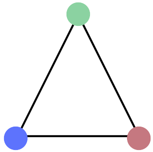

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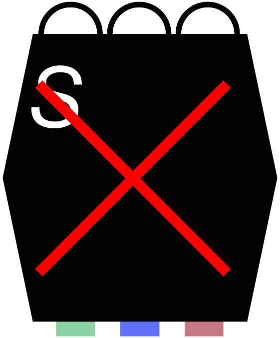

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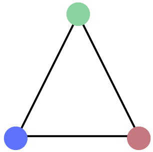

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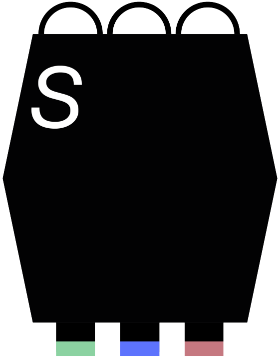

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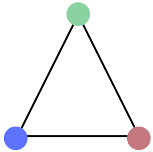

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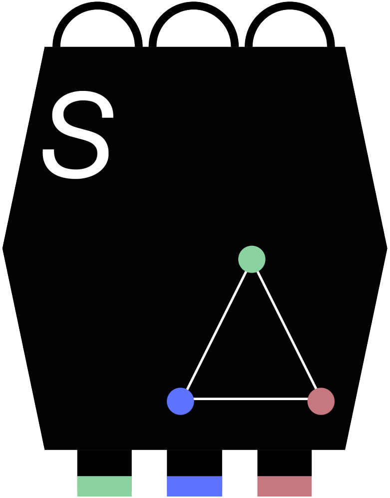

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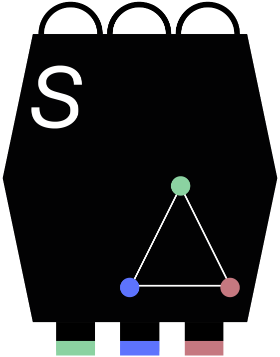

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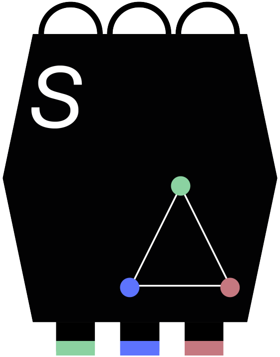

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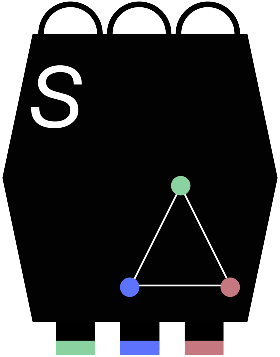

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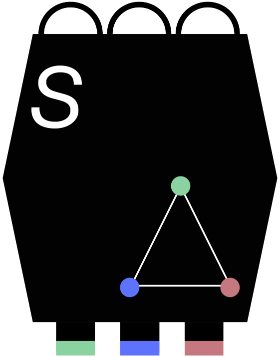

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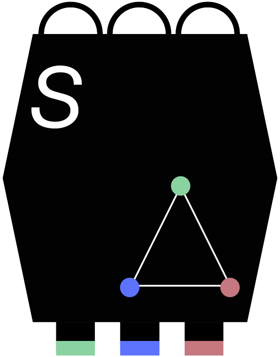

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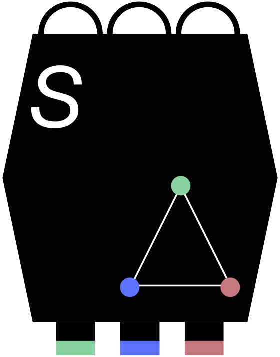

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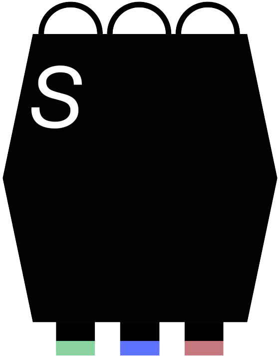

- Behaviour of system is described by measurement statistics

		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y				
y	z				
x	z				

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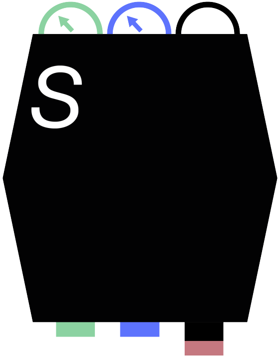

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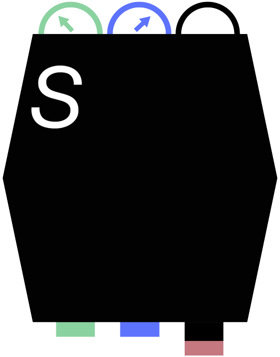

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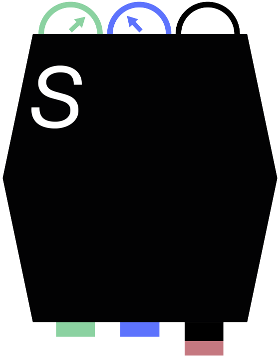

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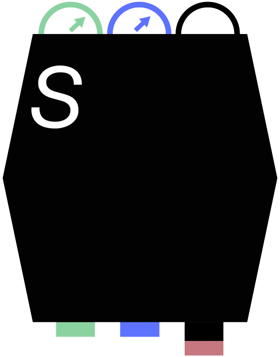

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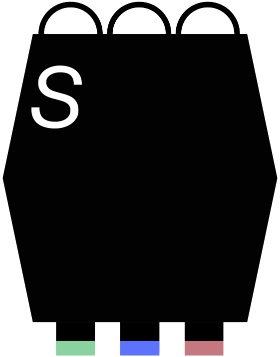

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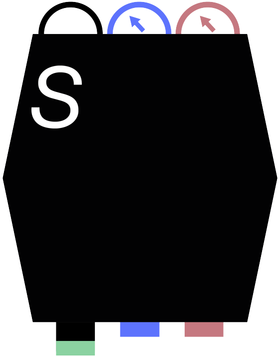

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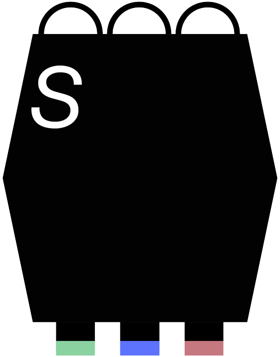

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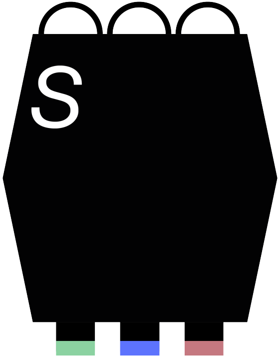

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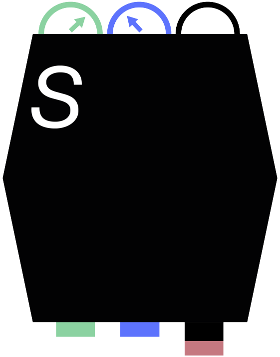
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

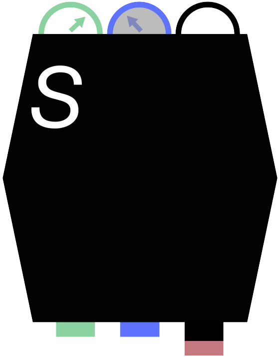
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$P(\mathsf{x}, \mathsf{y} \mapsto a, b)$$

Behaviour: empirical model



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x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
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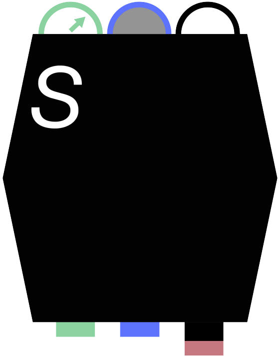

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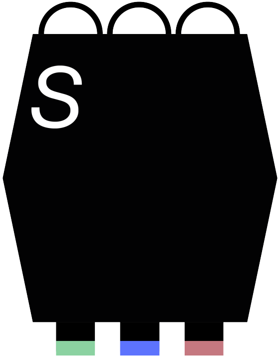
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_{b} P(\mathbf{x}, \mathbf{y} \mapsto a, b)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
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No-signalling / no-disturbance

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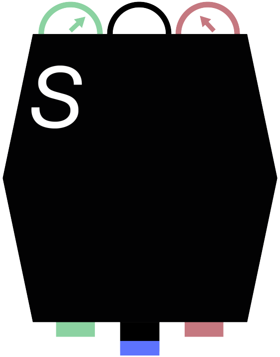


- Behaviour of system is described by measurement statistics

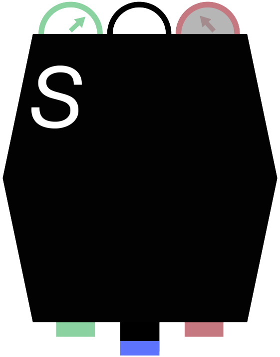
		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x},\mathsf{y} \mapsto a, b) \qquad\qquad P(\mathsf{x},\mathsf{z} \mapsto a, c)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

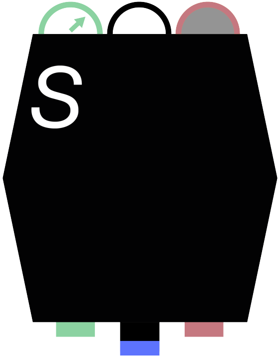
		(0,0)	(0,1)	(1,0)	(1,1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x}, \mathsf{y} \mapsto a, b) \qquad\qquad P(\mathsf{x}, \mathsf{z} \mapsto a, c)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

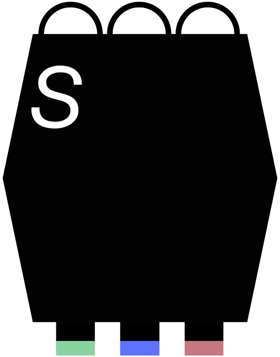
		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x}, \mathsf{y} \mapsto a, b) \qquad \sum_c P(\mathsf{x}, \mathsf{z} \mapsto a, c)$$

Behaviour: empirical model



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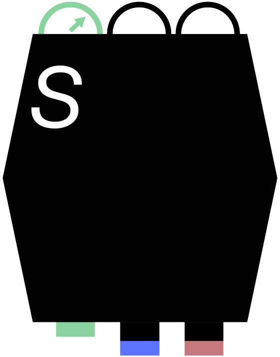
		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x},\mathsf{y} \mapsto a,b) \quad = \quad \sum_c P(\mathsf{x},\mathsf{z} \mapsto a,c)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
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No-signalling / no-disturbance

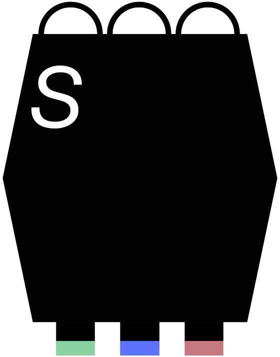
- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b) \qquad \sum_c P(\mathbf{x}, \mathbf{z} \mapsto a, c)$$

$$=$$

$$P(\mathbf{x} \mapsto a)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
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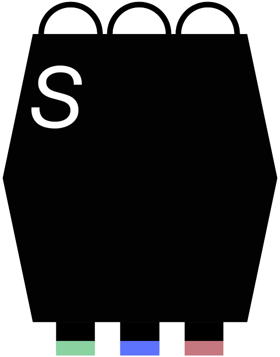
Behaviour: empirical model

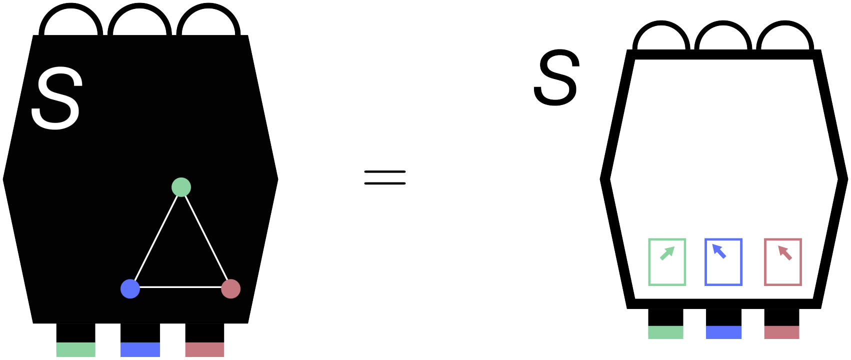


Empirical model $e : S$ is a family $\{e_\sigma\}_{\sigma \in \Sigma_S}$ where:

- e_σ is a probability distribution on the set of joint outcomes $\mathbf{O}_{S,\sigma} := \prod_{x \in \sigma} O_{S,x}$
- These satisfy **no-disturbance**:
  if $\tau \subset \sigma$, then $e_\sigma|_\tau = e_\tau$.

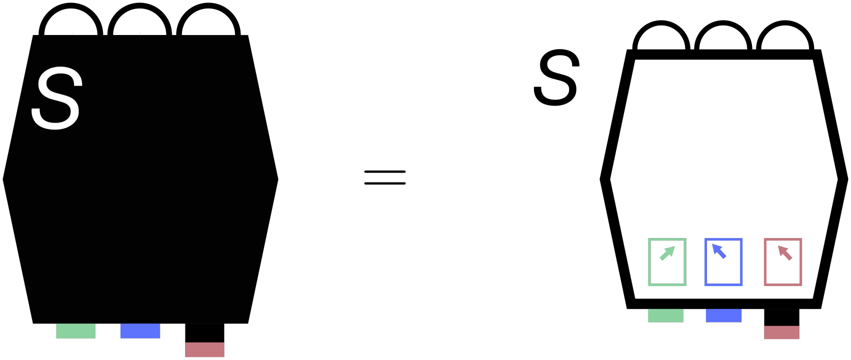
Contextuality

Deterministic model



Contextuality

Deterministic model



Contextuality

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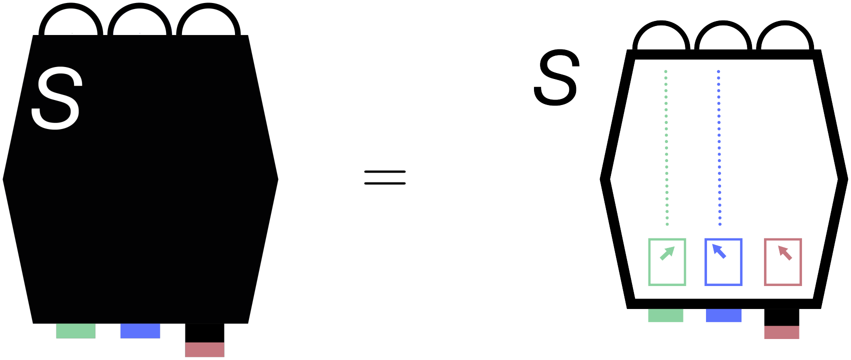
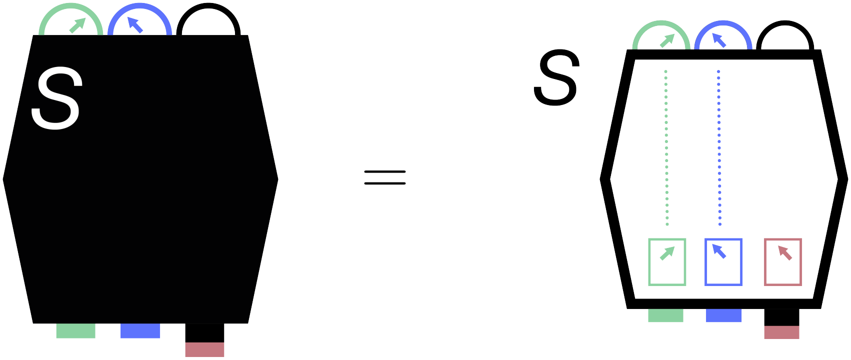


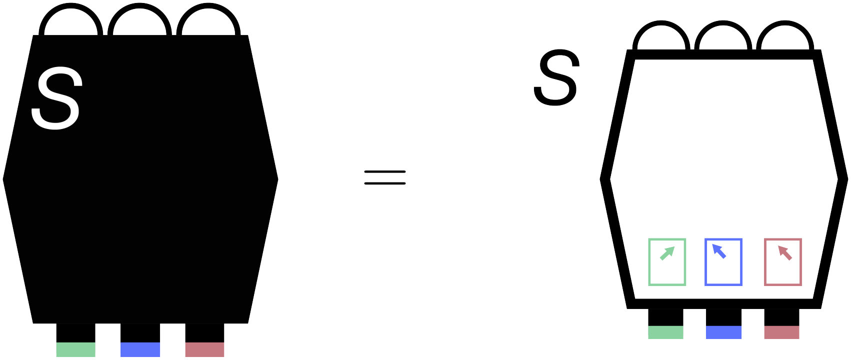
Contextuality

Deterministic model



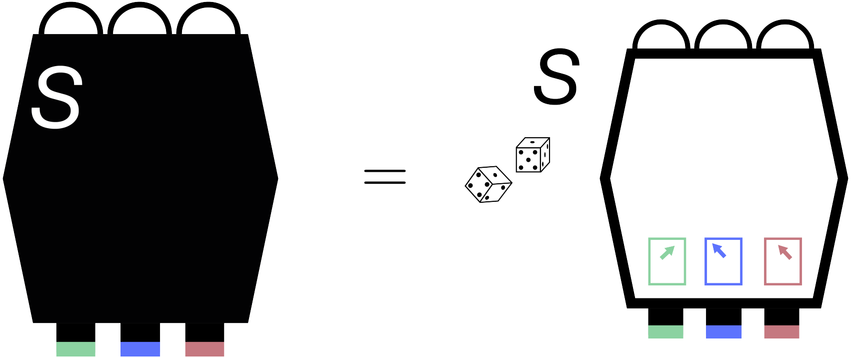
Contextuality

Deterministic model



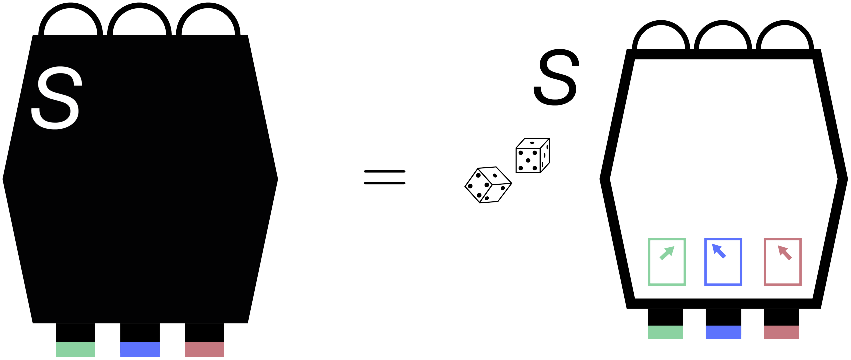
Contextuality

Non-contextual model



Contextuality

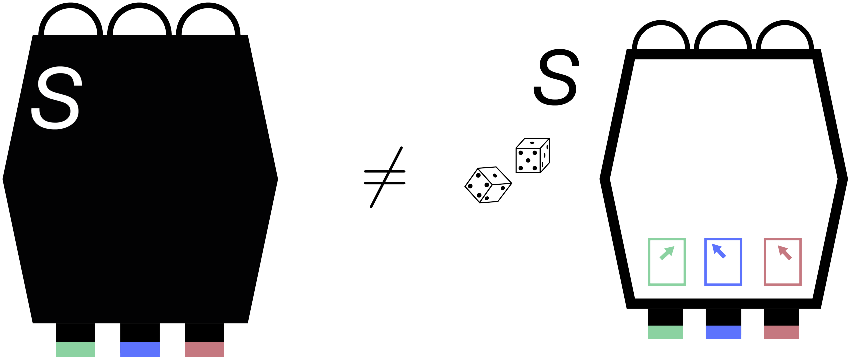
Non-contextual model



$\exists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Contextuality

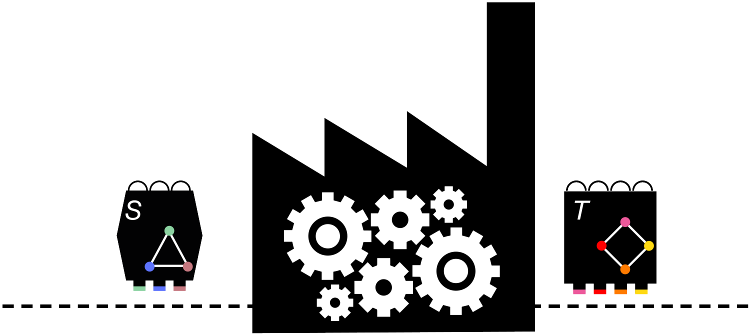
Contextual model



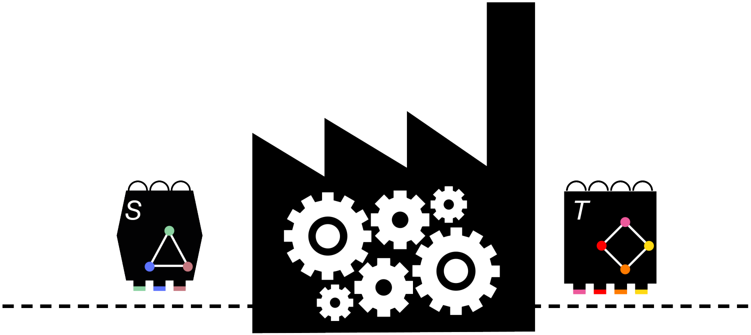
$\nexists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Resource theory of contextuality

Resource theories

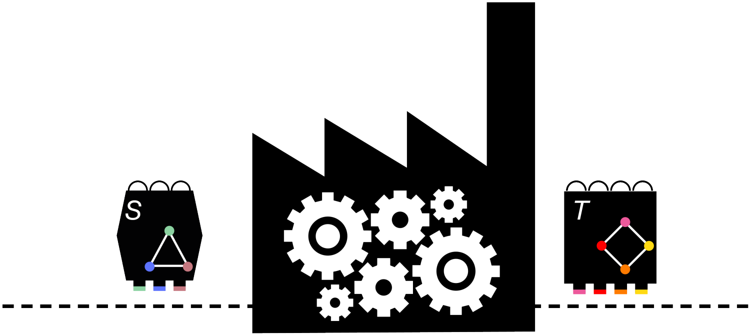


Resource theories



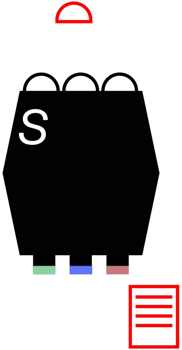
- Consider ‘free’ (i.e. classical) operations:

Resource theories



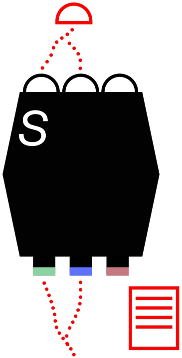
- Consider 'free' (i.e. classical) operations:
  (classical) procedures that use a box of type S to simulate a box of type T

Experiments and procedures



- An O-valued S-**experiment** is a protocol for an interaction with the box S producing a value in O:
  - which measurements to perform;
  - how to interpret their joint outcome into an outcome in O.

Experiments and procedures



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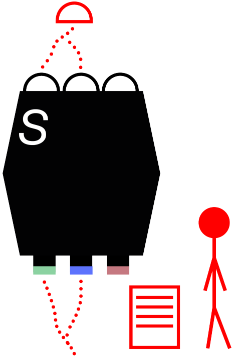

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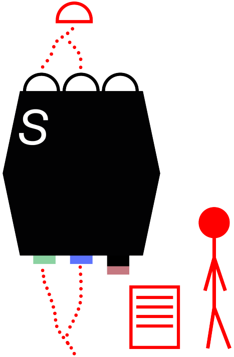

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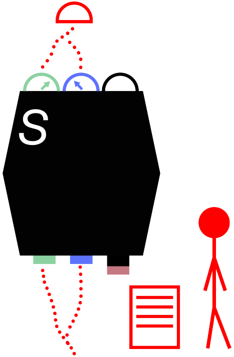

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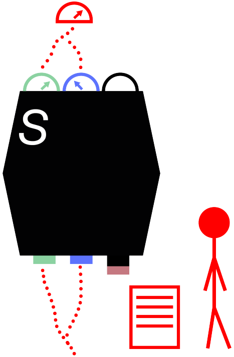

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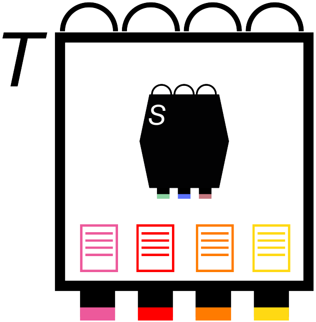
Experiments and procedures



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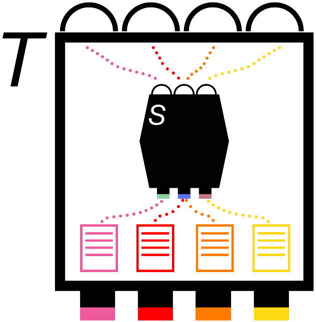
- A **deterministic procedure** $S \longrightarrow T$ specifies an S-experiment ($O_{T,x}$-valued) for each measurement x of T.

Experiments and procedures



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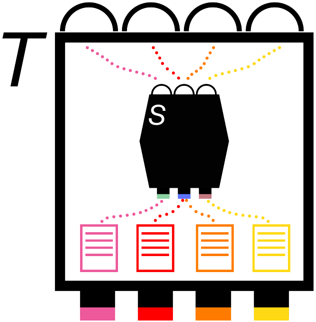
Experiments and procedures



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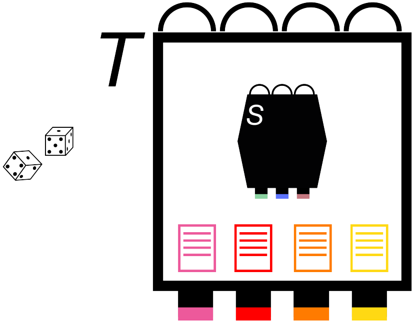
Experiments and procedures



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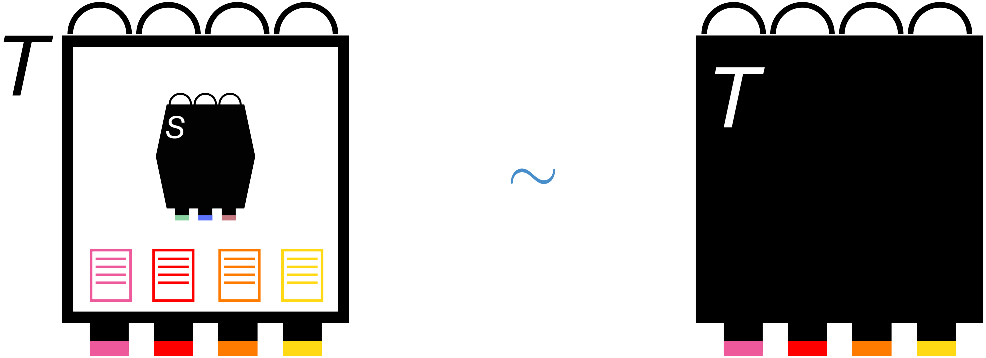
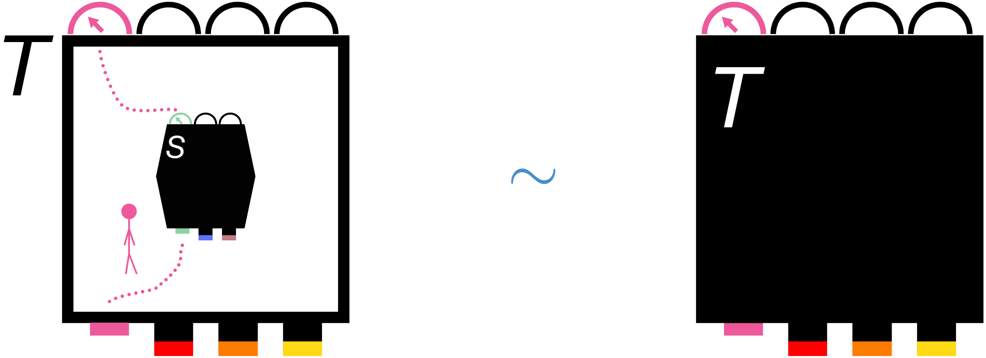
- A **classical procedure** is a probabilistic mixture of deterministic procedures.

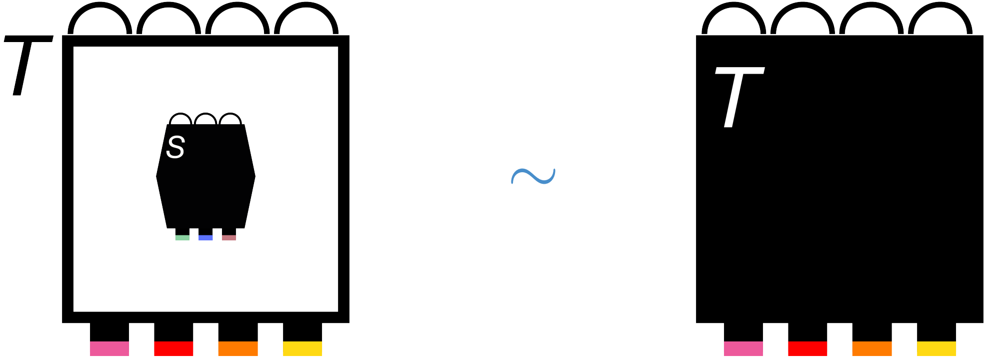
Classical procedures and simulations



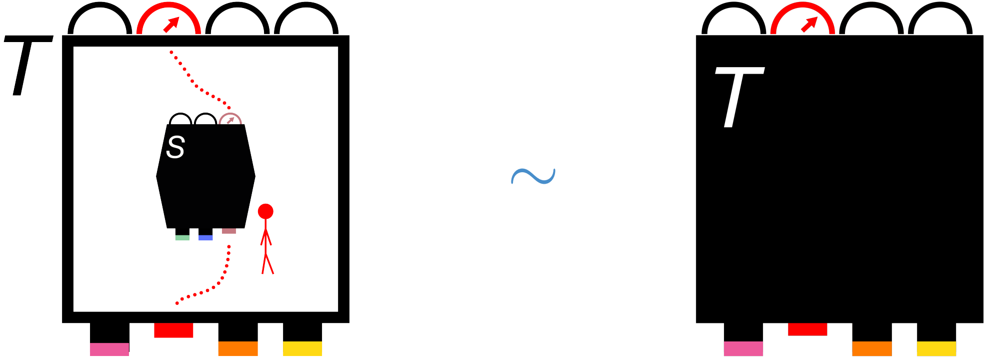
Classical procedures and simulations



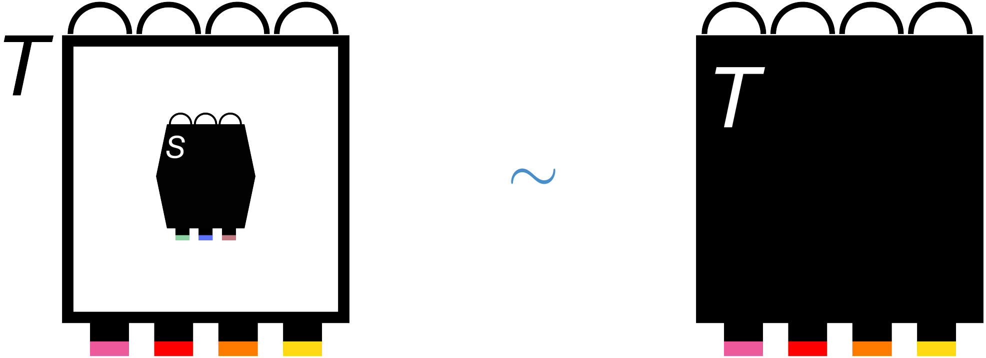
Classical procedures and simulations



Classical procedures and simulations



Classical procedures and simulations



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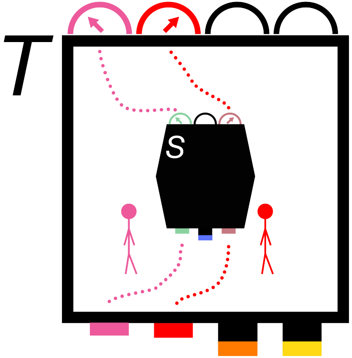

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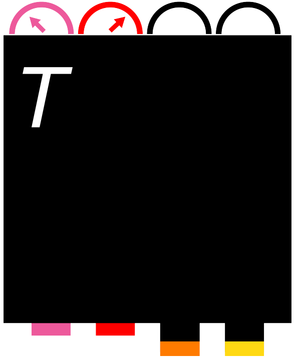
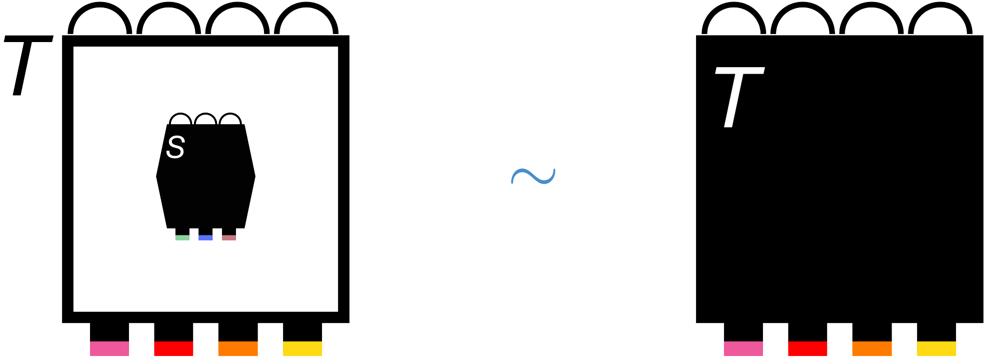


Classical procedures and simulations



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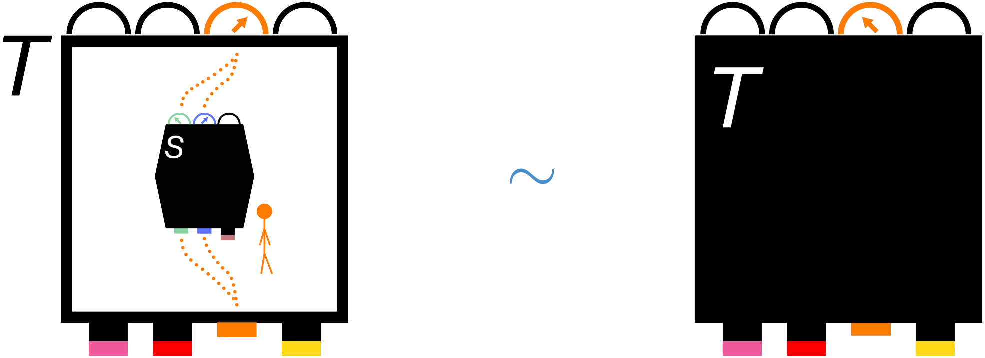

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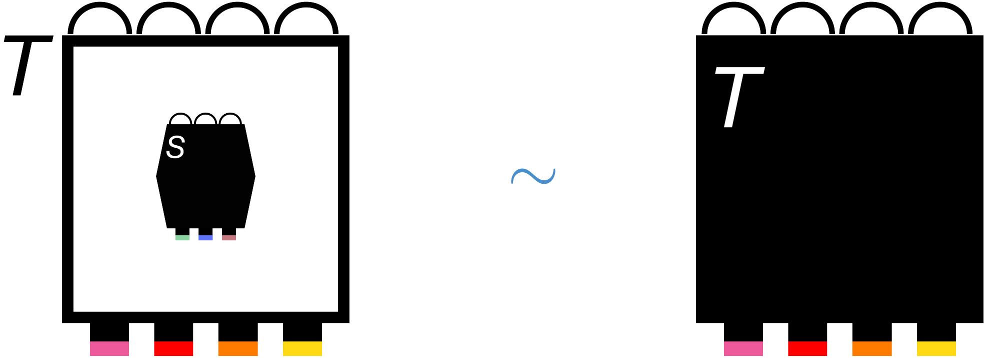

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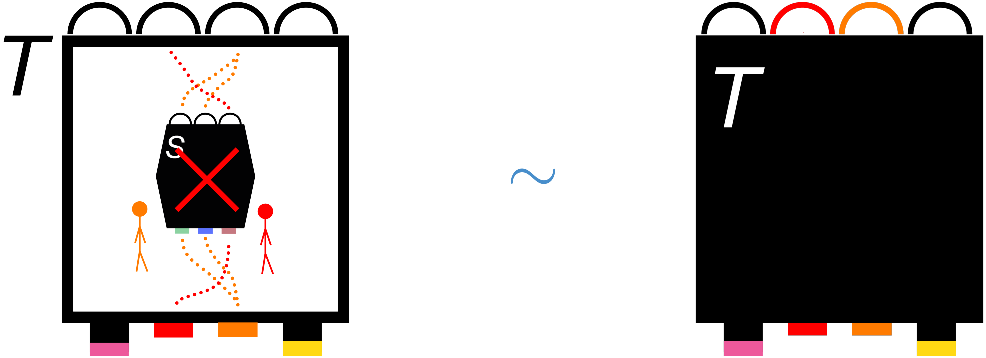

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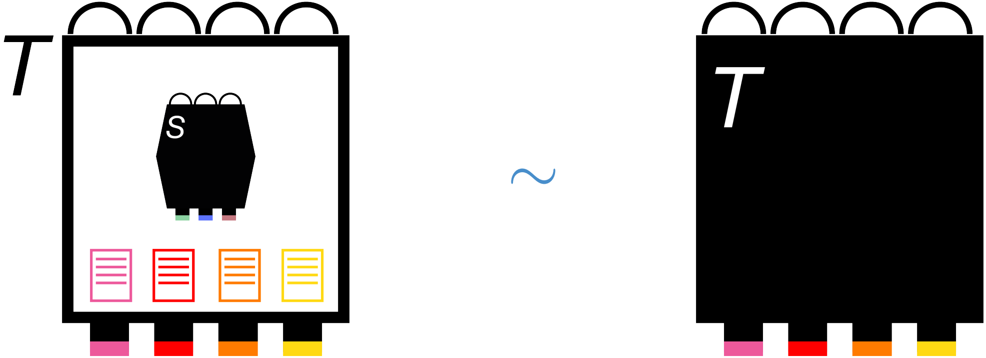

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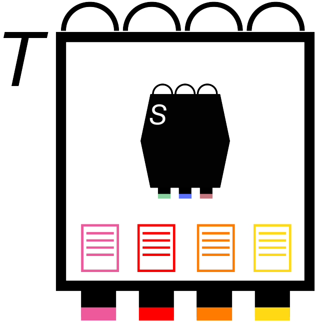
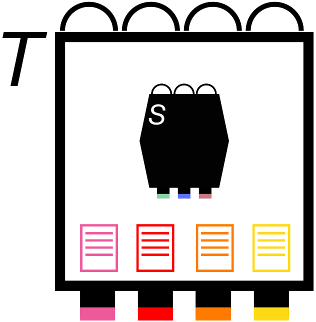


Deterministic procedure $f : S \longrightarrow T$ is $\langle \pi_f, \alpha_f \rangle$:

Classical procedures



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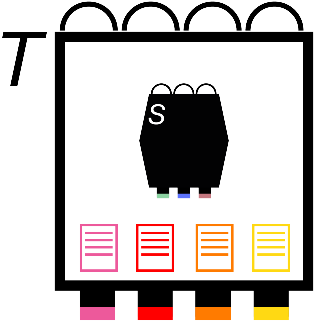

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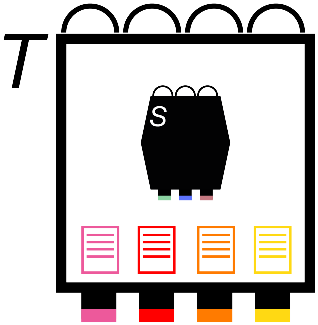

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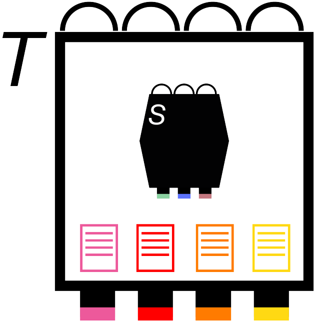

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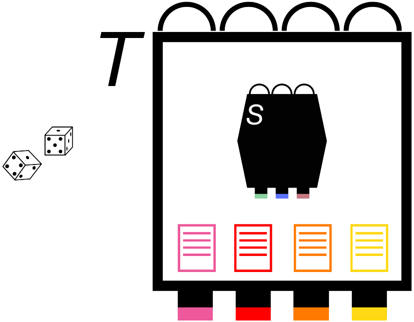

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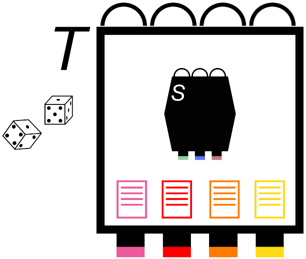
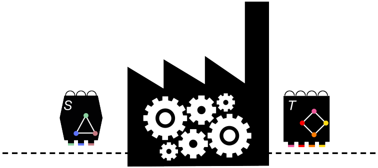
Probabilistic procedure $f : S \longrightarrow T$ is $f = \sum_i r_i f_i$ where $r_i \geq 0$, $\sum_i r_i = 1$, and $f_i : S \longrightarrow T$ deterministic procedures.

Classical simulations

- A classical procedure induces a (convex-preserving) map between empirical models:

$$f : S \longrightarrow T \qquad \rightsquigarrow \qquad \mathbf{Emp}(f) : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$$





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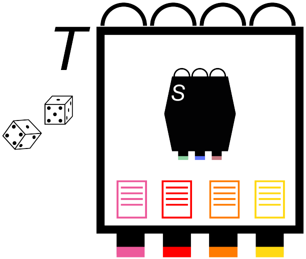

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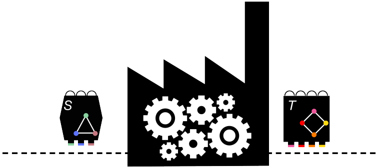

- Which black-box transformations arise in this fashion?

Characterising classical transformations

Given $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$, can it be realised by a classical procedure?
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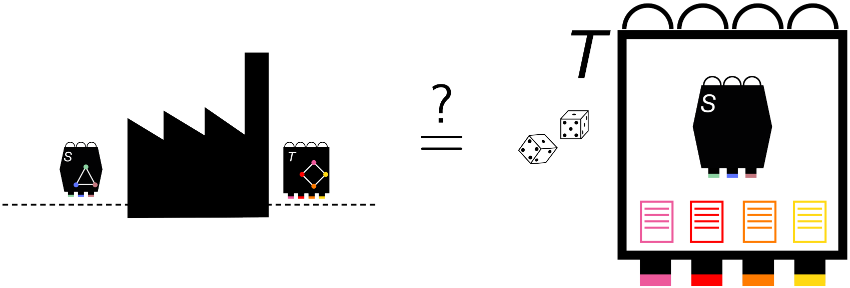

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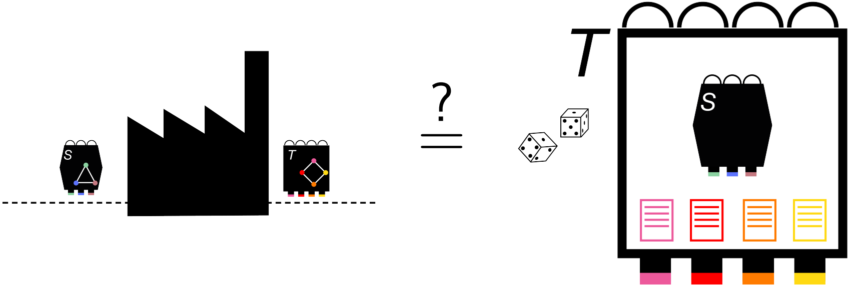

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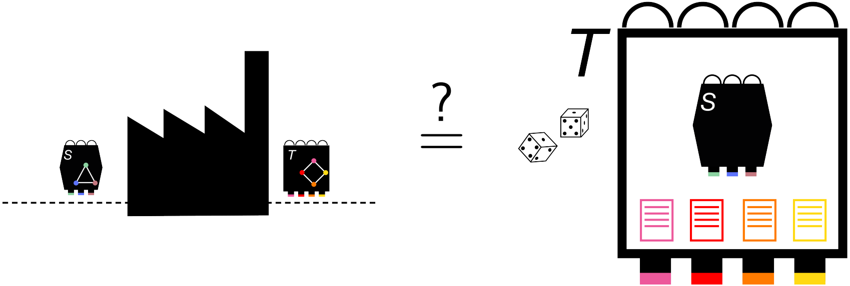

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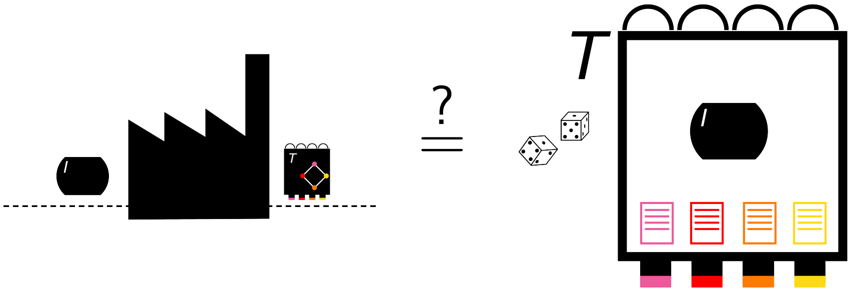

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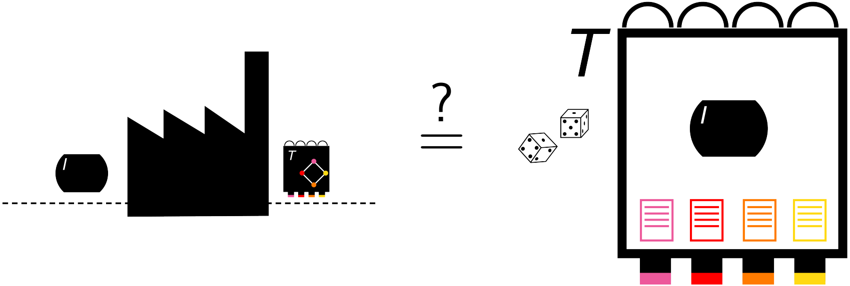

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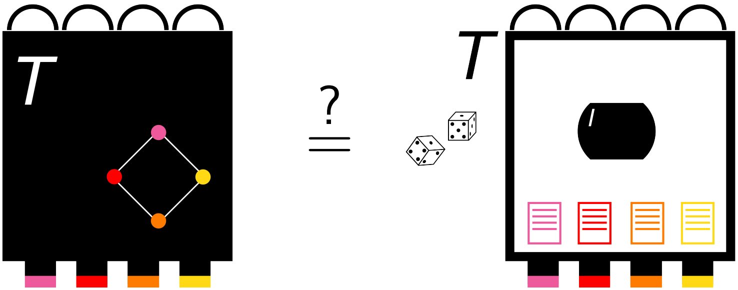

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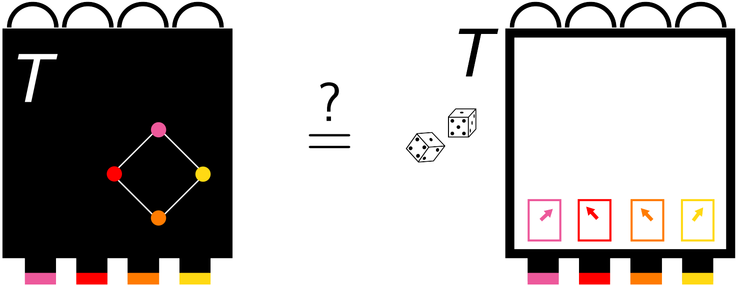

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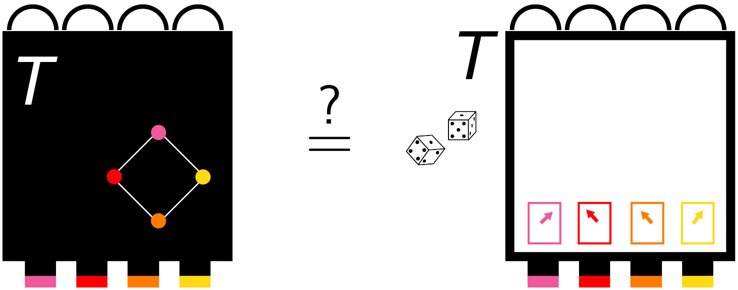
(Non-contextual models are those which can be simulated from nothing.)



From objects to morphisms . . .

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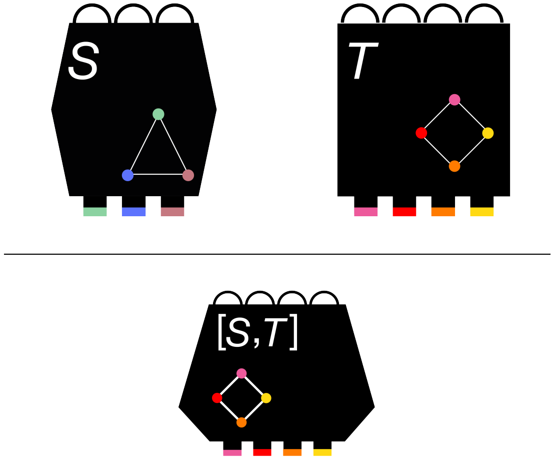

From two scenarios S and T, we build a new scenario $[S, T]$.

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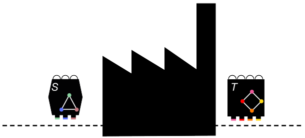

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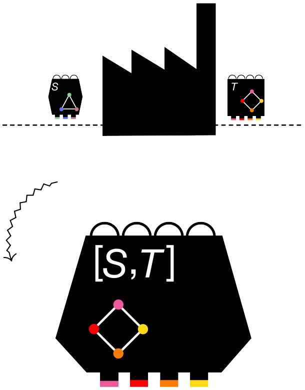

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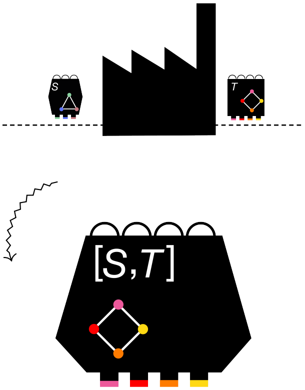

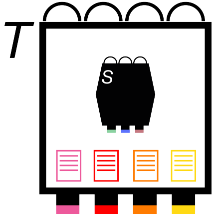

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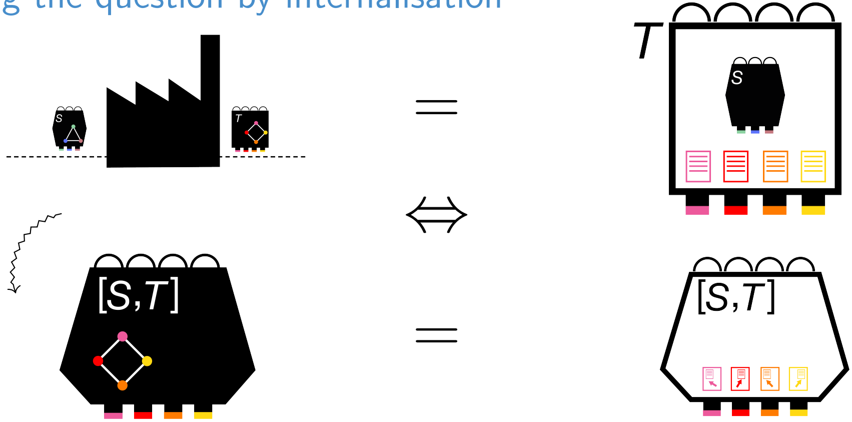

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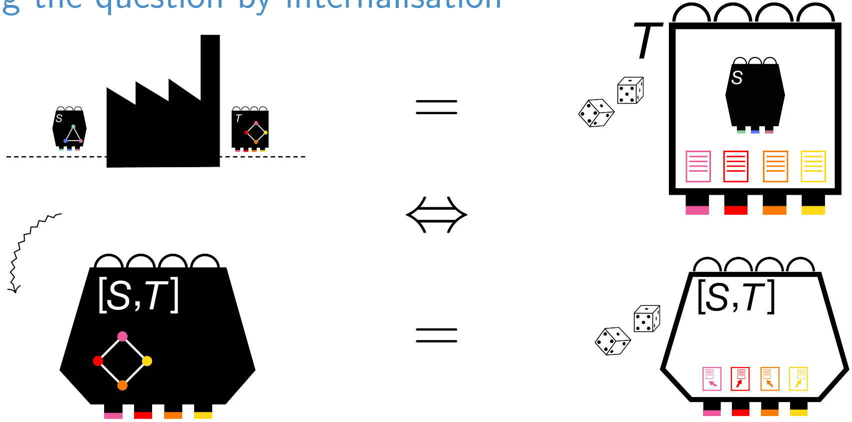

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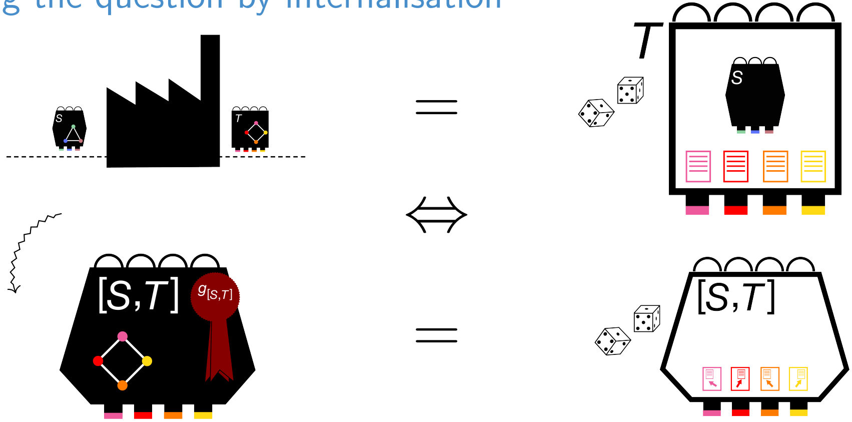

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F is realised by a deterministic procedure iff e_F is deterministic and satisfies $g_{[S,T]}$.

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- $f : S \longrightarrow 2$ to the single measurement two-outcome scenario is a **predicate**
  - It induces a map $EMP(S) \longrightarrow [0, 1]$ yielding the probability that it holds.

Adaptive simulations

Beyond non-adaptive simulations

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Formally, we construct a **comonad** MP on the category of empirical models, where MP($e \colon S$) is the model obtained by taking all measurement protocols over the given scenario.

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  - Two runs (of different protocols) are consistent if they agree on common measurements
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Proposition

MP defines a comonoidal comonad on the category of deterministic classical procedres (and therefore on the category of empirical models).

Roughly: comultiplication $\mathsf{MP}(S) \longrightarrow \mathsf{MP}^2(S)$ by "flattening", unit $\mathsf{MP}(S) \longrightarrow S$, and $\mathsf{MP}(S \otimes T) \longrightarrow \mathsf{MP}(S) \otimes \mathsf{MP}(T)$

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \longrightarrow e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, i.e. a map

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We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

The (partial algebraic) logical view

Algebra of predicates

- For simplicity, we make two restrictions to the kind of scenarios we consider:
  - all measurements are dichotomic: they have two outcomes 0 and 1 (think of true and false).
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- We can now think of predicates $f : S \longrightarrow 2$ as new measurements:
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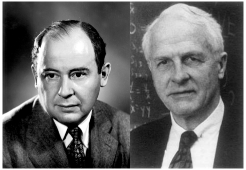
- $\rightsquigarrow$ **partial Boolean algebras**.

Quantum physics and logic



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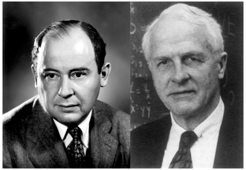

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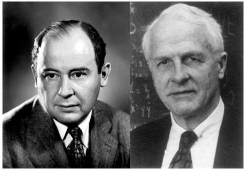

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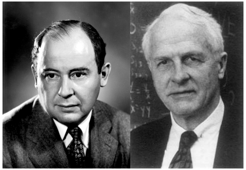

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- Only commuting measurements can be performed together.
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?

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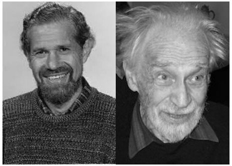

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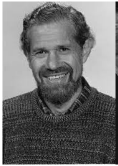
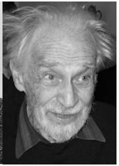

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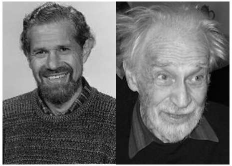

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

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satisfying the usual axioms: $\langle A, \vee, 0 \rangle$ and $\langle A, \wedge, 1 \rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap \rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

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Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined.
This gives a category **pBA**.

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Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

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- No assignment of truth values to all propositions which respects logical operations on jointly testable propositions.

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- **BA** is a full subcategory of **pBA**.
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We could say that such a diagram is "implicitly contradictory", and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

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But where do states come in?

States

Definition

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Proposition

States can be characterised as the maps $\nu : A \longrightarrow [0,1]$ such that, for every Boolean subalgebra B of A, the restriction of ν to B is a finitely additive probability measure on B.

We can define a state $\nu : A \longrightarrow [0,1]$ to be **probabilistically non-contextual** if ν extends to $\mathcal{C}(A)$; that is, there is a state $\hat{\nu} : \mathcal{C}(A) \longrightarrow [0,1]$ such that $\nu = \hat{\nu} \circ \eta$.

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By the universal property of $\mathcal{C}(A)$, this is equivalent to asking that there is some Boolean algebra B, morphism $h : A \longrightarrow B$, and state $\hat{\nu} : B \longrightarrow [0,1]$ such that $\nu = \hat{\nu} \circ \eta$.

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Note that if A is K-S, $\mathcal{C}(A) = \mathbf{1}$, and there is no state on $\mathbf{1}$.

Connection to the sheaf-theoretic approach

Free partial Boolean algebra on a reflexive graph $(X_S, \frown)$ (a 'graphical' measurement scenario).

- Generators $G := \{\imath(x) \mid x \in X\}$.
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- The reason is that new compatibilities arise!

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Given two graphical measurement scenarios:

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- More general resolutions of identity: LEP

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Relativising Logical Bell inequalities

A simple observation

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$$1 = \mathrm{Prob}\left(\neg \bigwedge \phi_i\right) = \mathrm{Prob}\left(\bigvee \neg\phi_i\right) \leq \sum_{i=1}^{N} \mathrm{Prob}\left(\neg\phi_i\right)$$

A simple observation

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- Hence,

$$\sum_{i=1}^{N} p_i \leq N - 1 \quad .$$

Questions...

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