A structural account of composition methods in logic

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Resources in Computation, UCL

Motivation

Our setting: $\mathcal{R}(\sigma) = \sigma$-structures and their homomorphisms

Mostowski's theorem

$$A_1 \equiv_{FO} B_1 \text{ and } A_2 \equiv_{FO} B_2 \quad \text{implies} \quad A_1 \times A_2 \equiv_{FO} B_1 \times B_2$$

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Feferman–Vaught's theorem

$$A_1 \equiv_{FO} B_1 \text{ and } A_2 \equiv_{FO} B_2 \quad \text{implies} \quad A_1 \mathbin{\dot\cup} A_2 \equiv_{FO} B_1 \mathbin{\dot\cup} B_2$$

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⋮

For $\tau \subseteq \sigma$, reduct operation $\mathrm{fg}_\tau \colon \mathcal{R}(\sigma) \to \mathcal{R}(\tau)$

$$A \equiv_{FO(\sigma)} B \quad \text{implies} \quad \mathrm{fg}_\tau(A) \equiv_{FO(\tau)} \mathrm{fg}_\tau(B)$$

General statement

Given an operation

$$H: \; \mathcal{R}(\sigma_1) \times \cdots \times \mathcal{R}(\sigma_n) \to \mathcal{R}(\sigma_{n+1})$$

and logics

$$L_1 \text{ in signature } \sigma_1$$
$$\vdots$$
$$L_{n+1} \text{ in signature } \sigma_{n+1}$$

we wish to know if, for every $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$,

$$A_i \equiv_{L_i} B_i \;\; (\forall i) \quad \text{implies} \quad H(A_1, \ldots, A_n) \equiv_{L_{n+1}} H(B_1, \ldots, B_n).$$

Question

Is there a principal way to prove these theorems?

- Without knowing anything about the operation?
- Without knowing anything about the logic fragments?
- Without having to deal with syntax?

Positive existential Ehrenfeucht–Fraïssé games semantically

Proposition

The following are equivalent:

- *Duplicator has a winning strategy in the k-round* **existential** *Ehrenfeucht–Fraïssé game from A to B.*
- $A \Rrightarrow_{\exists^+ FO_k} B$, *i.e.* $\forall$ **positive existential** φ *of qrank* $\leq k$,

$$A \models \varphi \quad \textit{implies} \quad B \models \varphi$$

allowed: $\exists, \wedge, \vee$
banned: $\forall, \neg$

- $A \Rrightarrow_{\exists^+ \mathbb{E}_k} B$, *i.e. there exists a homomorphism* $\mathbb{E}_k(A) \to B$.

Intuitively

$$\mathbb{E}_k(A) = \text{a } \sigma\text{-structure of Spoiler's plays on } A \text{ \underline{in } k \text{ rounds}}$$

The universe of $\mathbb{E}_k(A)$ is

$$\Big\{ [a_1, \ldots, a_n] \;\Big|\; a_i \in A,\; n \leq k \Big\}$$

and, for $w_1 = [a_{1,1}, \ldots, a_{1,n_1}],\ \ldots\ , w_u = [a_{u,1}, \ldots, a_{u,n_u}]$,

$$\begin{aligned}(w_1,\ \ldots\ , w_u) \in R^{\mathbb{E}_k(A)} \quad &\Longleftrightarrow \quad (a_{1,n_1},\ \ldots\ , a_{u,n_u}) \in R^A \\ &\qquad \text{and} \quad w_i \sqsubseteq w_j \text{ or } w_j \sqsubseteq w_i \quad (\forall i, j)\end{aligned}$$

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$$\text{and} \quad w_i \sqsubseteq w_j \text{ or } w_j \sqsubseteq w_i \quad (\forall i, j)$$

Theorem

$(\mathbb{E}_k, \varepsilon, \overline{(-)})$ *is a comonad.*

where

$$\varepsilon_A \colon \mathbb{E}_k(A) \to A, \quad [a_1, \dots, a_n] \mapsto a_n$$

and

$$\overline{(-)}\colon\ (\mathbb{E}_k(A) \xrightarrow{f} B) \ \mapsto\ (\mathbb{E}_k(A) \xrightarrow{\overline{f}} \mathbb{E}_k(B))$$

More game comonads

We have

$$A \Rrightarrow_{\exists^{+}\mathbb{C}} B \quad \iff \quad A \Rrightarrow_{\exists^{+}\mathcal{L}} B$$

for

- the E–F comonad $\mathbb{E}_k$ and qrank $\leq k$ fragment
- the Pebbling comonad $\mathbb{P}_k$ and k-variable fragment
- the modal comonad $\mathbb{M}_k$ and modal depth $\leq k$ fragment
- the Pebble-Relation comonad $\mathbb{PR}_k$ and the restricted conjunction k-variable fragment
- the Hella comonad $\mathbb{H}_k$ and the generalised quantifier k-variable extension
- the guarded comonad $\mathbb{G}_k$ and the k-guarded fragment

$\vdots$

Positive existential fragments and FVM theorems

Test case: How can we prove this?

$$A \Rrightarrow_{\exists^+ FO_k} B \quad \text{implies} \quad \mathrm{fg}_\tau(A) \Rrightarrow_{\exists^+ FO_k} \mathrm{fg}_\tau(B)$$

i.e.

$$\text{from} \quad \mathbb{E}_k(A) \xrightarrow{f} B \quad \text{produce} \quad \mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{f'} \mathrm{fg}_\tau(B)$$

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Observe, we have

$$\mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{\kappa_A} \mathrm{fg}_\tau(\mathbb{E}_k(A)), \quad w \mapsto w$$

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therefore

$$\mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{\kappa_A} \mathrm{fg}_\tau(\mathbb{E}_k(A)) \xrightarrow{\mathrm{fg}_\tau(f)} \mathrm{fg}_\tau(B)$$

Categorical positive existential FVM theorem

Theorem

Assume we have

- *a functor* $H\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{C}_{n+1}$
- *comonads* $\mathbb{C}_1, \ldots, \mathbb{C}_{n+1}$ *on* $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$
- *and morphisms*

$$\mathbb{C}_{n+1}(H(A_1, \ldots, A_n)) \xrightarrow{\kappa} H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n))$$

Then,

$$A_i \Rrightarrow_{\exists^+\mathbb{C}_i} B_i \qquad \text{for } i = 1, \ldots, n$$

implies

$$H(A_1, \ldots, A_n) \;\Rrightarrow_{\exists^+\mathbb{C}_{n+1}}\; H(B_1, \ldots, B_n)$$

Counting logics and comonads

Proposition

For finite A, B, the following are equivalent:

- *Duplicator has a winning strategy in the* **bijective** *k-round Ehrenfeucht–Fraïssé game from A to B.*
- $A \equiv_{\#FO_k} B$, *i.e.* $\forall$ **counting** φ *of qrank* $\leq k$,

$$A \models \varphi \quad \text{if and only if} \quad B \models \varphi.$$

using: $\exists^{\geq n} x$

- $A \equiv_{\#\mathbb{E}_k} B$, *i.e. there exist homomorphisms*

$$\mathbb{E}_k(A) \xrightarrow{f} B \text{ and } \mathbb{E}_k(B) \xrightarrow{g} A$$

s.t.

$$\mathbb{E}_k(A) \xrightarrow{\overline{f}} \mathbb{E}_k(B) \xrightarrow{\overline{g}} \mathbb{E}_k(A) = id$$

$$\mathbb{E}_k(B) \xrightarrow{\overline{g}} \mathbb{E}_k(A) \xrightarrow{\overline{f}} \mathbb{E}_k(B) = id$$

Counting logics and FVM theorems

Why

$$A \equiv_{\#FO_k} B \quad \text{implies} \quad \mathrm{fg}_\tau(A) \equiv_{\#FO_k} \mathrm{fg}_\tau(B) \text{ ?}$$

I.e. given

$$\mathbb{E}_k(A) \xrightarrow{f} B \text{ and } \mathbb{E}_k(B) \xrightarrow{g} A \quad \text{s.t.} \quad \overline{f} \circ \overline{g} = \mathrm{id} \text{ and } \overline{g} \circ \overline{f} = \mathrm{id}$$

find

$$\mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{f'} \mathrm{fg}_\tau(B) \text{ and } \mathbb{E}_k(\mathrm{fg}_\tau(B)) \xrightarrow{g'} \mathrm{fg}_\tau(A) \quad \text{s.t.} \quad \ldots$$

Counting logics and FVM theorems

Why

$$A \equiv_{\#FO_k} B \quad \text{implies} \quad \mathrm{fg}_\tau(A) \equiv_{\#FO_k} \mathrm{fg}_\tau(B) \text{ ?}$$

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find

$$\mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{f'} \mathrm{fg}_\tau(B) \text{ and } \mathbb{E}_k(\mathrm{fg}_\tau(B)) \xrightarrow{g'} \mathrm{fg}_\tau(A) \quad \text{s.t.} \quad \ldots$$

recall

$$f' = \mathbb{E}_k(\mathrm{fg}_\tau(A)) \xrightarrow{\kappa_A} \mathrm{fg}_\tau(\mathbb{E}_k(A)) \xrightarrow{\mathrm{fg}_\tau(f)} \mathrm{fg}_\tau(B)$$

$$g' = \mathbb{E}_k(\mathrm{fg}_\tau(B)) \xrightarrow{\kappa_B} \mathrm{fg}_\tau(\mathbb{E}_k(B)) \xrightarrow{\mathrm{fg}_\tau(g)} \mathrm{fg}_\tau(A)$$

Will they do?

Counting logics and FVM theorems, II

κ is a **Kleisli law:**

$$
\begin{array}{ccc}
\mathbb{E}_k(\mathrm{fg}_\tau(A)) & \xrightarrow{\kappa} & \mathrm{fg}_\tau(\mathbb{E}_k(A)) \\
& {\scriptstyle \varepsilon_{\mathrm{fg}_\tau(A)}} \searrow \quad \swarrow {\scriptstyle \mathrm{fg}_\tau(\varepsilon_{A_i})} & \\
& \mathrm{fg}_\tau(A) &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{E}_k(\mathrm{fg}_\tau(A)) & \xrightarrow{\kappa_A} & \mathrm{fg}_\tau(\mathbb{E}_k(A)) \\
{\scriptstyle \overline{f'}} \downarrow & & \downarrow {\scriptstyle \mathrm{fg}_\tau(\overline{f})} \\
\mathbb{E}_k(\mathrm{fg}_\tau(B)) & \xrightarrow{\kappa_B} & \mathrm{fg}_\tau(\mathbb{E}_k(B))
\end{array}
$$

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Theorem

Assume we have a Kleisli law

$$\mathbb{C}_{n+1}(H(A_1, \ldots, A_n)) \to H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n))$$

Then

$$A_i \equiv_{\#\mathbb{C}_i} B_i \, (\forall i) \;\; \textit{implies} \;\; H(A_1, \ldots, A_n) \equiv_{\#\mathbb{C}_{n+1}} H(B_1, \ldots, B_n)$$

Full logics and comonads

Proposition

The following are equivalent:

- *Duplicator has a winning strategy in the (weak) k-round Ehrenfeucht–Fraïssé game from A to B.*
- $A \equiv_{\mathrm{FO}_k^-} B$, *i.e.* $\forall$ *first-order* φ *of qrank* $\leq k$,
$$A \models \varphi \quad \textit{implies} \quad B \models \varphi.$$
- $A \equiv_{\mathbb{E}_k} B$, *i.e. there exist homomorphisms*
$$R \xrightarrow{f} \mathbb{E}_k(A) \qquad R \xrightarrow{g} \mathbb{E}_k(B) \qquad R \xrightarrow{\rho} \mathbb{E}_k(R)$$
such that
  - ρ *is an* $\mathbb{E}_k$*-coalgebra*
  - f, g *are "open pathwise-embedding"* $\mathbb{E}_k$*-coalgebra morphisms*

Disclaimer: FO_k^- is without equality!

Full logics and FVM theorems

Theorem

Assume we have a Kleisli law κ, as earlier, that $\mathbb{C}_{n+1}$ and H preserve embeddings, and any

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\quad\quad\quad f \quad\quad\quad\quad} & & H(A_1, \ldots, A_n) \\ {\scriptstyle P}\downarrow & & & & \downarrow{\scriptstyle H(\alpha_1, \ldots, \alpha_n)} \\ \mathbb{C}_{n+1}(P) & \xrightarrow{\mathbb{C}_{n+1}(f)} & \mathbb{C}_{n+1}H(A_1, \ldots, A_n) & \xrightarrow{\kappa} & H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n)) \end{array}$$

where (P, π) is a path and (A_i, α_i) coalgebras, has a minimal decomposition through "subpaths" of (A_i, α_i).

Then

$$A_i \equiv_{\mathbb{C}_i} B_i \, (\forall i) \quad \textit{implies} \quad H(A_1, \ldots, A_n) \equiv_{\mathbb{C}_{n+1}} H(B_1, \ldots, B_n)$$

Extensions

Set σ^I to be σ with a fresh binary $I(\cdot,\cdot)$ and

$$\mathtt{t}^I\colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$$

interpreting I as equality. Then

$$A \equiv_{FO_k} B \quad \text{iff} \quad \mathtt{t}^I(A) \equiv_{\mathbb{E}_k} \mathtt{t}^I(B)$$

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$$A \equiv_{FO_k} B \quad \text{iff} \quad \mathtt{t}^I(A) \equiv_{\mathbb{E}_k} \mathtt{t}^I(B)$$

Observation 1

For FVM theorems, it is enough if
$H:\ \mathcal{R}(\sigma_1) \times \cdots \times \mathcal{R}(\sigma_n) \to \mathcal{R}(\sigma_{n+1})$ commutes with $\mathtt{t}^I$.

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Observation 2

Not specific to $\mathtt{t}^I$, the same holds for any $\mathtt{t} \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^+)$.

Examples

- Logical equivalence in the restricted conjunction fragment of 3-variable counting logic implies cospectrality.
- Arbitrary coproducts and the k-variable resp. qrank $\leq k$ logics.
- Coproducts and FOL extended with connectivity relation.
- Products in any category where images of paths are paths
- Therefore, products for modal logics with global modalities.
- Coproducts and MSOL.

⋮

Thank you!

Remark for category-theorists

Our axioms of Kleisli laws equivalent to the usual ones, and generalise comonad morphisms.

With $\mathbb{C}_{n+1}$ preserving embeddings and $\mathcal{C}_{n+1}$ sufficiently complete H lifts to $\widehat{H}$:

$$
\begin{array}{ccc}
\mathcal{C}_1 \times \cdots \times \mathcal{C}_n & \xrightarrow{F^{\mathbb{C}}} & \texttt{CoAlg}(\mathbb{C}_1) \times \cdots \times \texttt{CoAlg}(\mathbb{C}_n) \\
{\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{H}} \\
\mathcal{C}_{n+1} & \xrightarrow{F^{\mathbb{C}_{n+1}}} & \texttt{CoAlg}(\mathbb{C}_{n+1})
\end{array}
$$

The last assumption in the last FVM corresponds to the fact that $\widehat{H}$ is a local relative right adjoint. (see Weber 2004, and e.g. Altenkirch et al 2010)