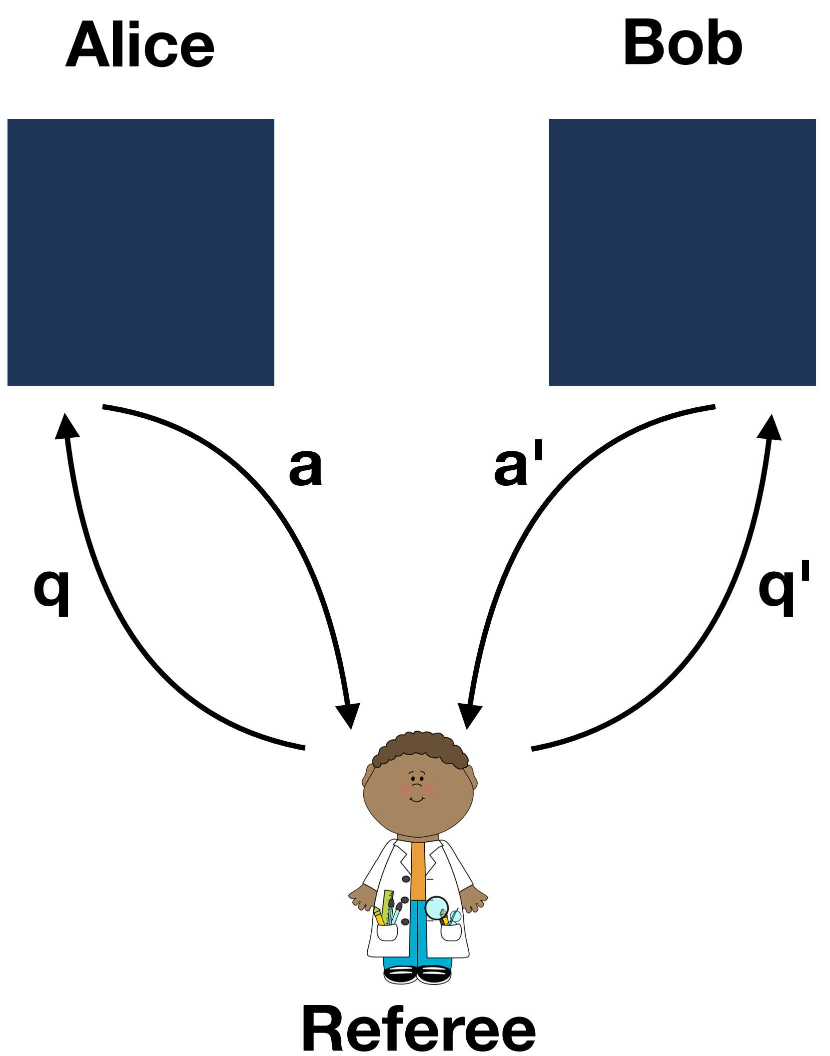
Results about mixed distributive laws motivated by considerations in logic and non-local games

Non-local games [1]



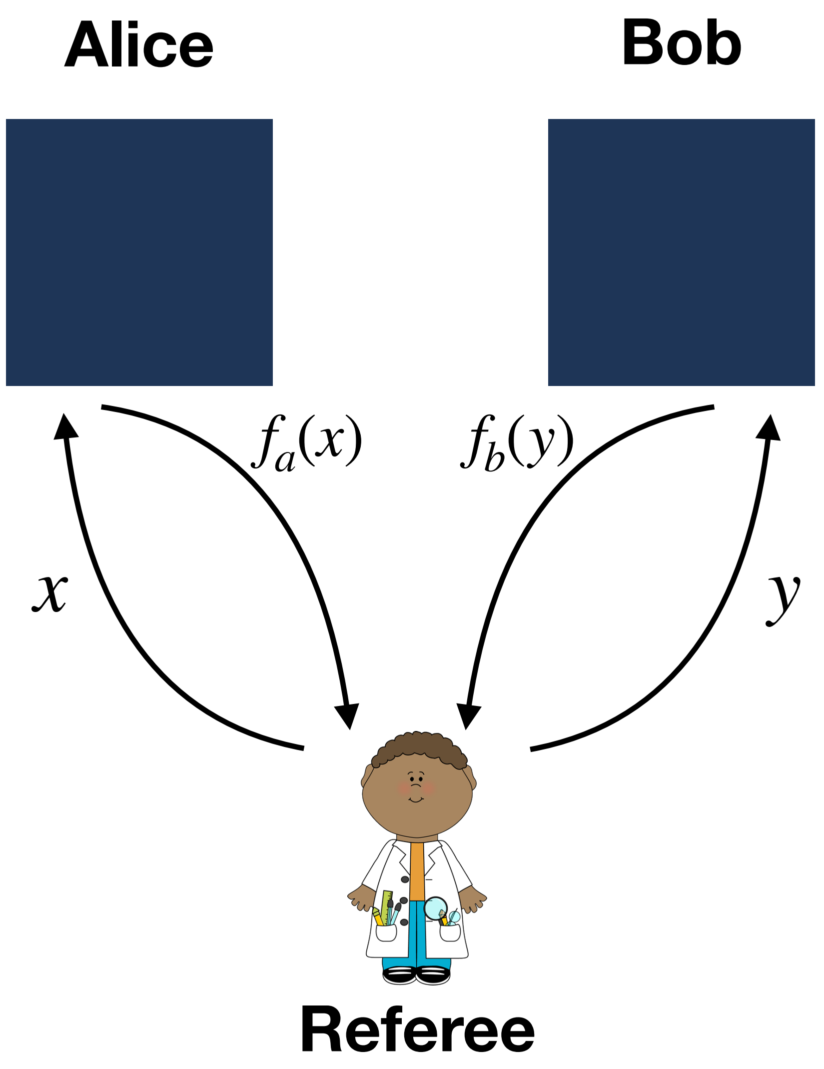
1. Referee sends a question to each player

2. Players answer **without communicating**

3. Win if answers satisfy some predefined conditions.

[1] Mathematics of quantum entanglement via nonlocal games: https://qmath.ku.dk/events/conferences/quantum-entanglement/

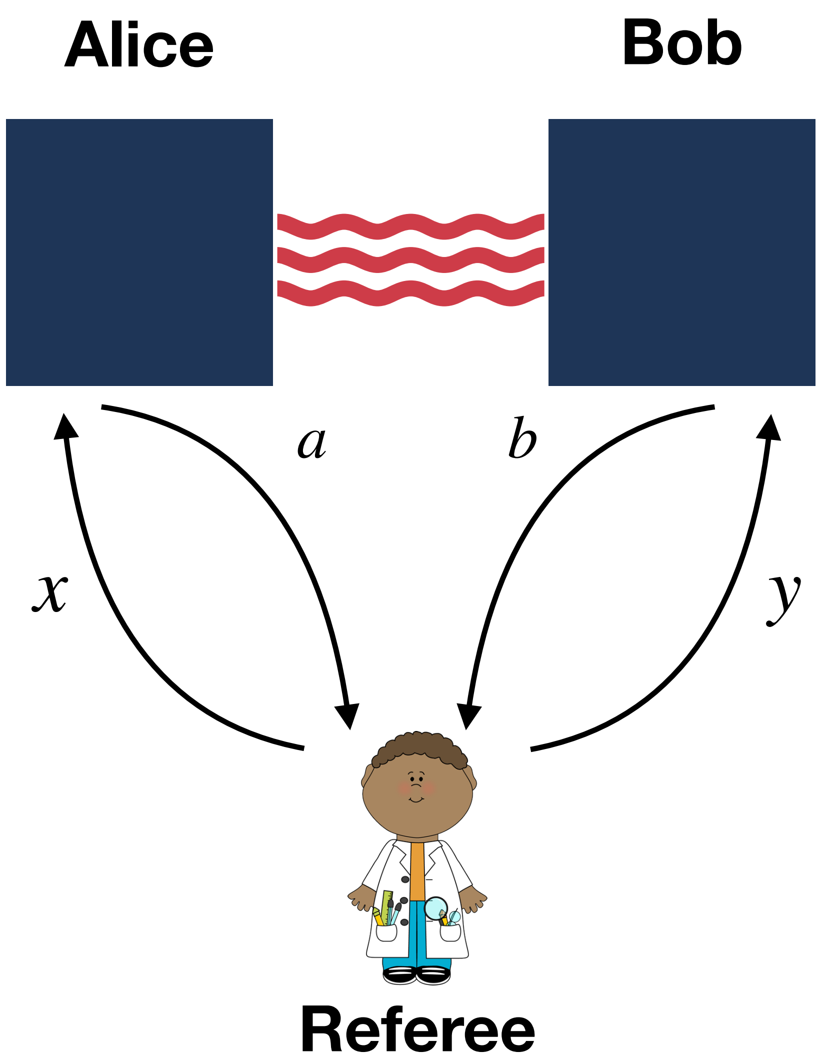
Classical Strategies



- Deterministic functions f_a and f_b.

$$p(f_a(x), f_b(y) \,|\, x, y) = 1$$

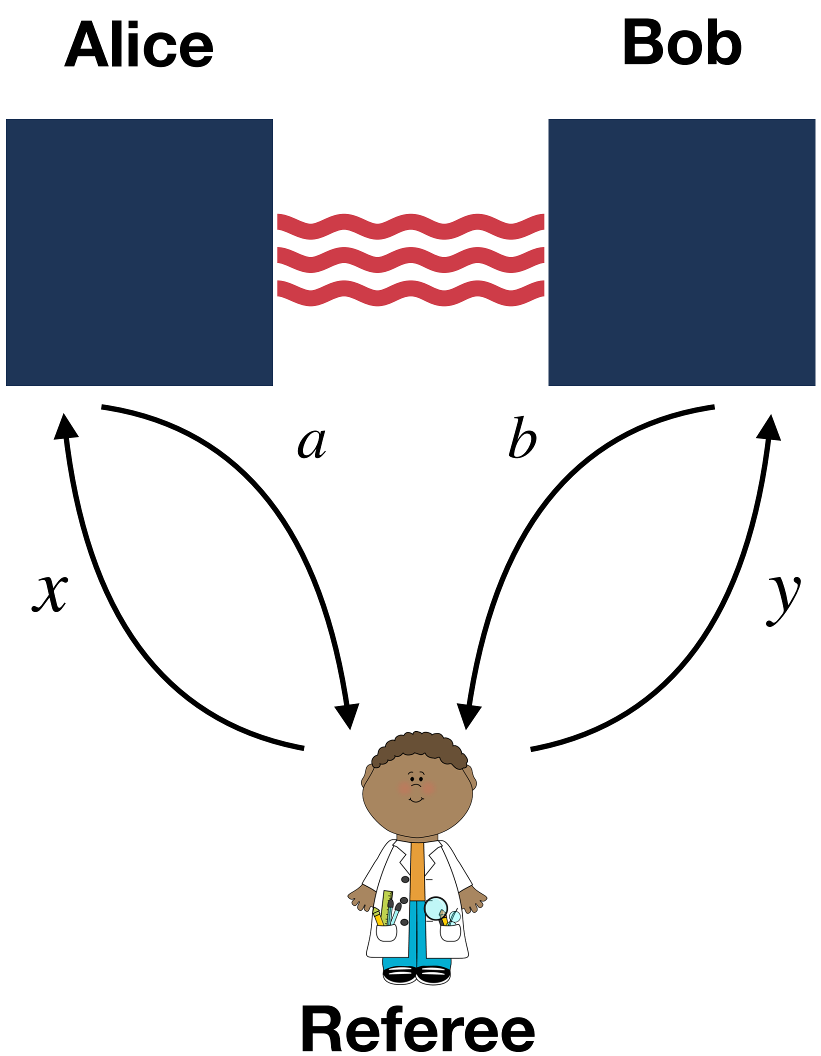
Quantum Tensor Strategies



- Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$
- Shared entangled state $\psi \in \mathscr{H}_A \otimes \mathscr{H}_B$
- For any inputs x, y, POVMs $\{A_{x,a}\}_a$, $\{B_{x,b}\}_b$ acting on $\mathscr{H}_A$ and $\mathscr{H}_B$

$$p(a, b \,|\, x, y) = \psi^{\dagger} A_{x,a} \otimes B_{y,b} \psi$$

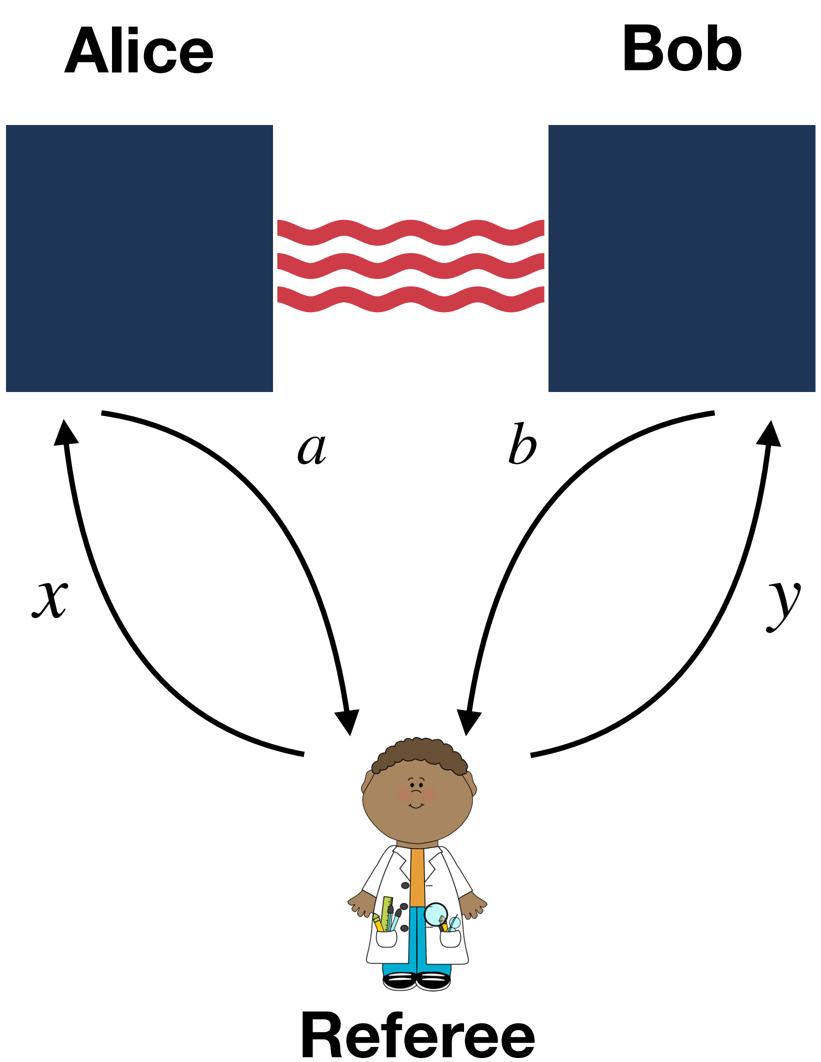
Quantum Commuting Strategies



- Hilbert space $\mathscr{H}$
- Shared entangled state $\psi \in \mathscr{H}$
- For any inputs x, y, POVMs $\{A_{x,a}\}_a$, $\{B_{x,b}\}_b$, acting on $\mathscr{H}$
- $A_{x,a}$ and $B_{y,b}$ commute for all x, a, y, b.

$$p(a, b \,|\, x, y) = \psi^\dagger A_{x,a} B_{y,b} \psi$$

Non-Signalling Strategies

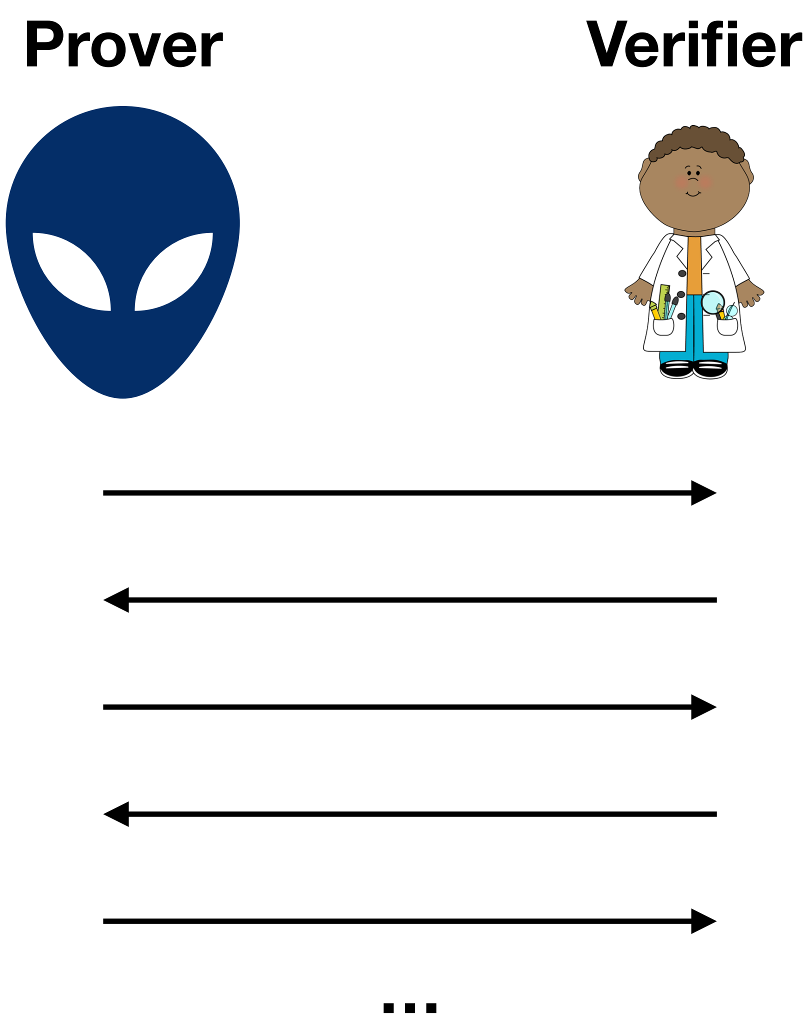


- Any strategy where:

$$\sum_{y_b} p(y_a, y_b \,|\, x_a, x_b) = \sum_{y_b} p(y_a, y_b \,|\, x_a, x_b') \,\forall x_a, y_a, x_b, x_b'$$

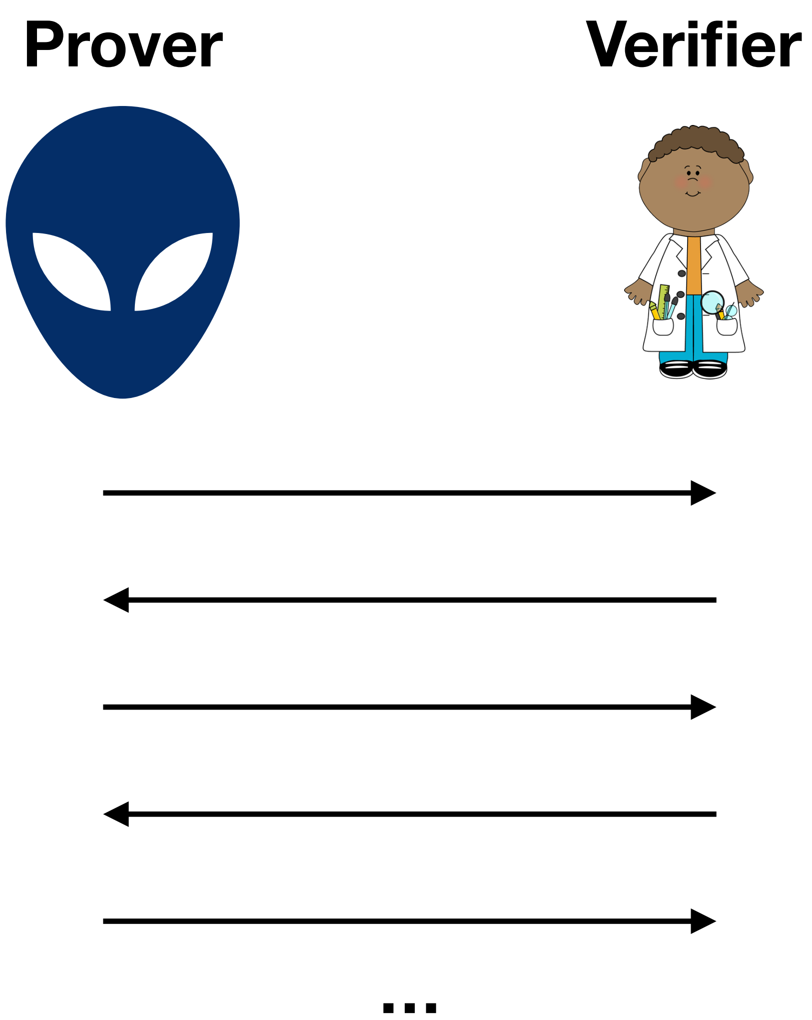
- Most general class of strategies with no communication.

Interlude: Interactive Proofs



- **All-powerful** prover exchanges messages with a **computationally limited** verifier.
- Prover tries to convince verifier that some string belongs to a language.
- **Soundness:** Cannot convince verifier of a false statement
- **Completeness:** Can convince verifier of a true statement.

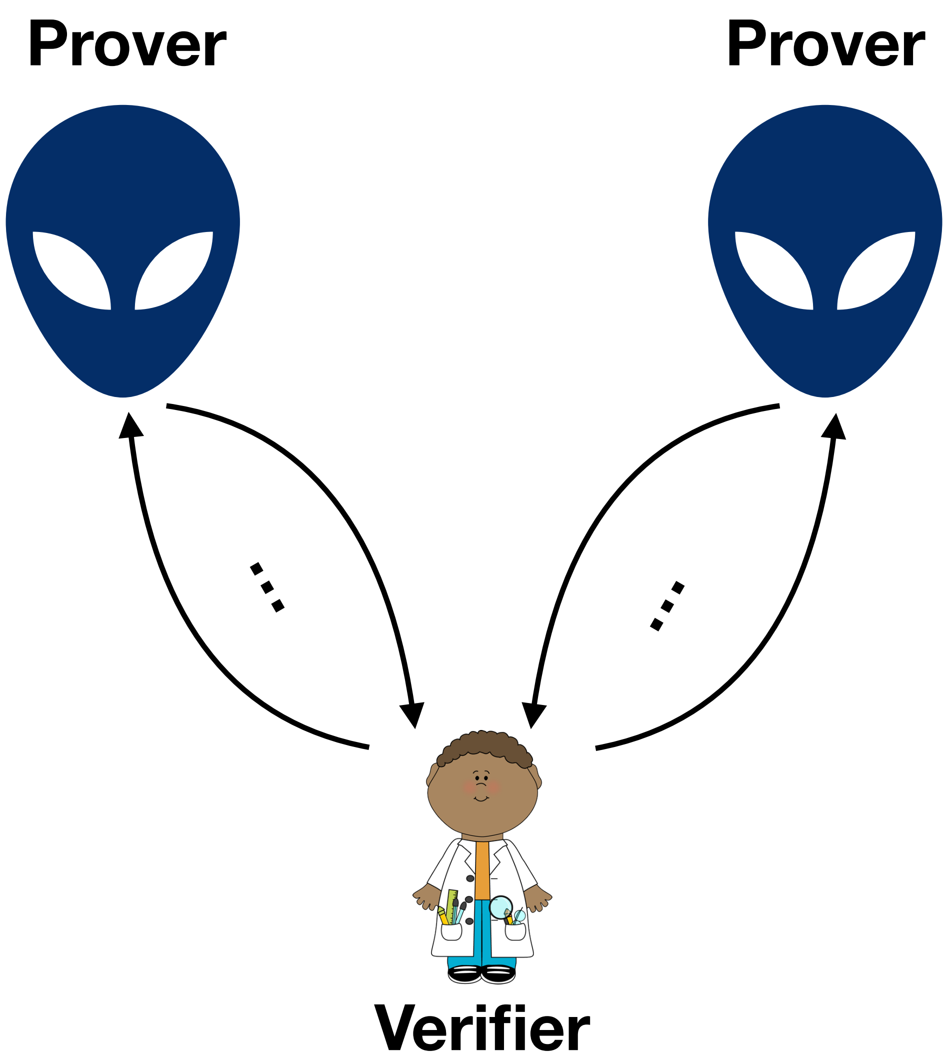
Interlude: Interactive Proofs



- Different interactive proof systems give rise to different complexity classes
- Single message exchange and PTIME verifier → NP
- Polynomially many messages and BPP verifier → IP
- Polynomially many quantum messages and BQP verifier → QIP

IP = QIP = PSPACE

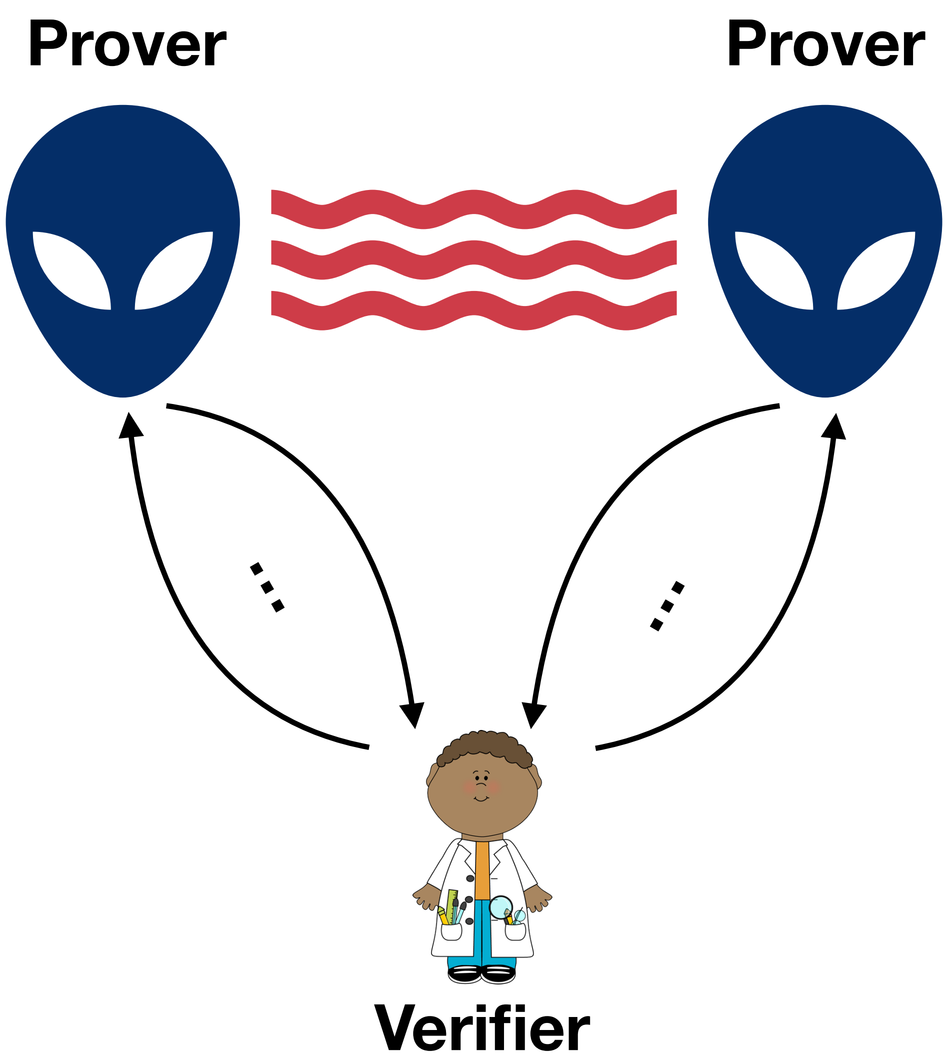
Multi-prover Interactive Proofs



- Polynomially many messages and BPP verifier $\rightarrow$ MIP
- Polynomially many quantum messages and BQP verifier $\rightarrow$ QMIP

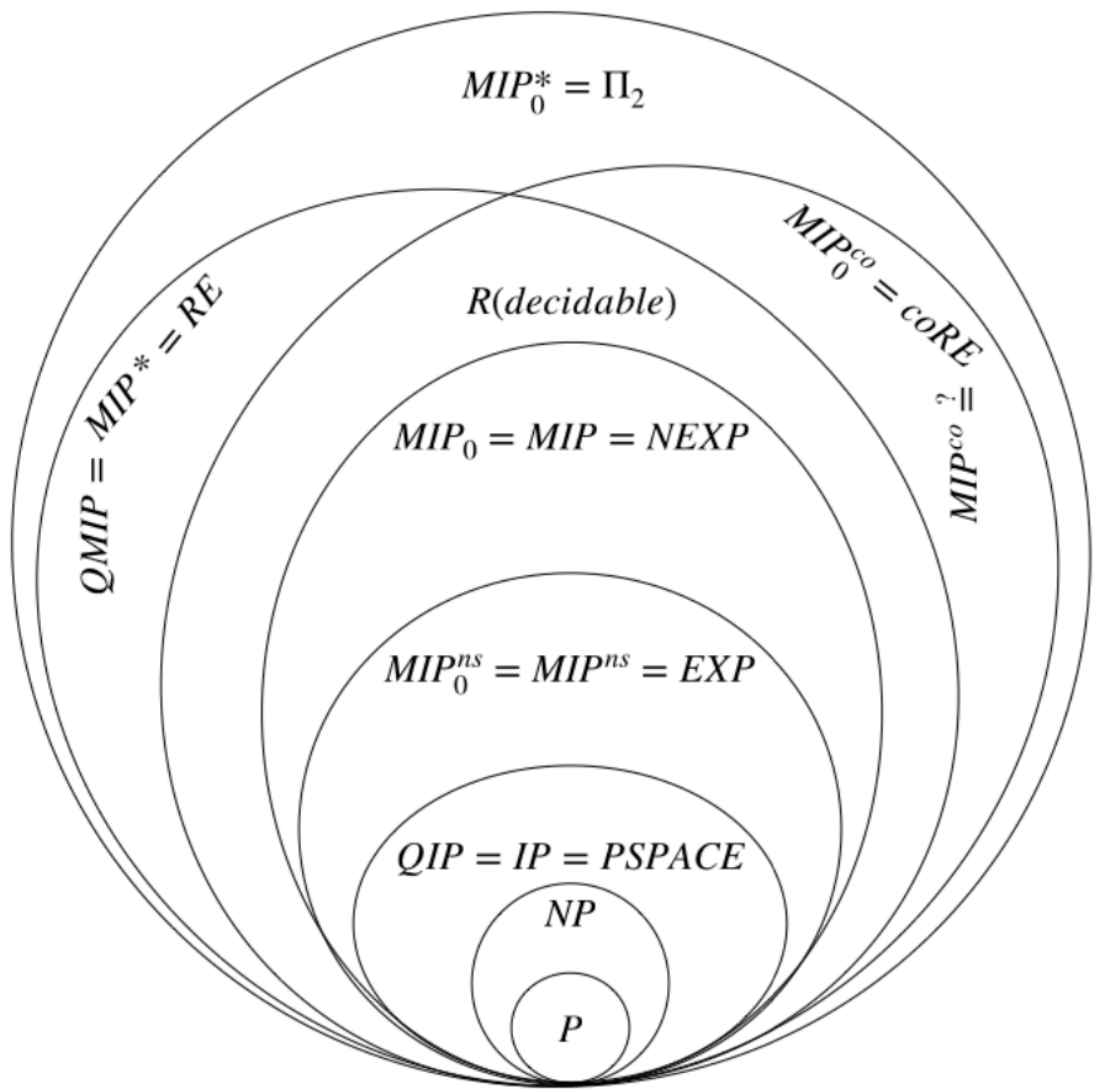
MIP = QMIP = NEXP

Multi-prover Interactive Proofs

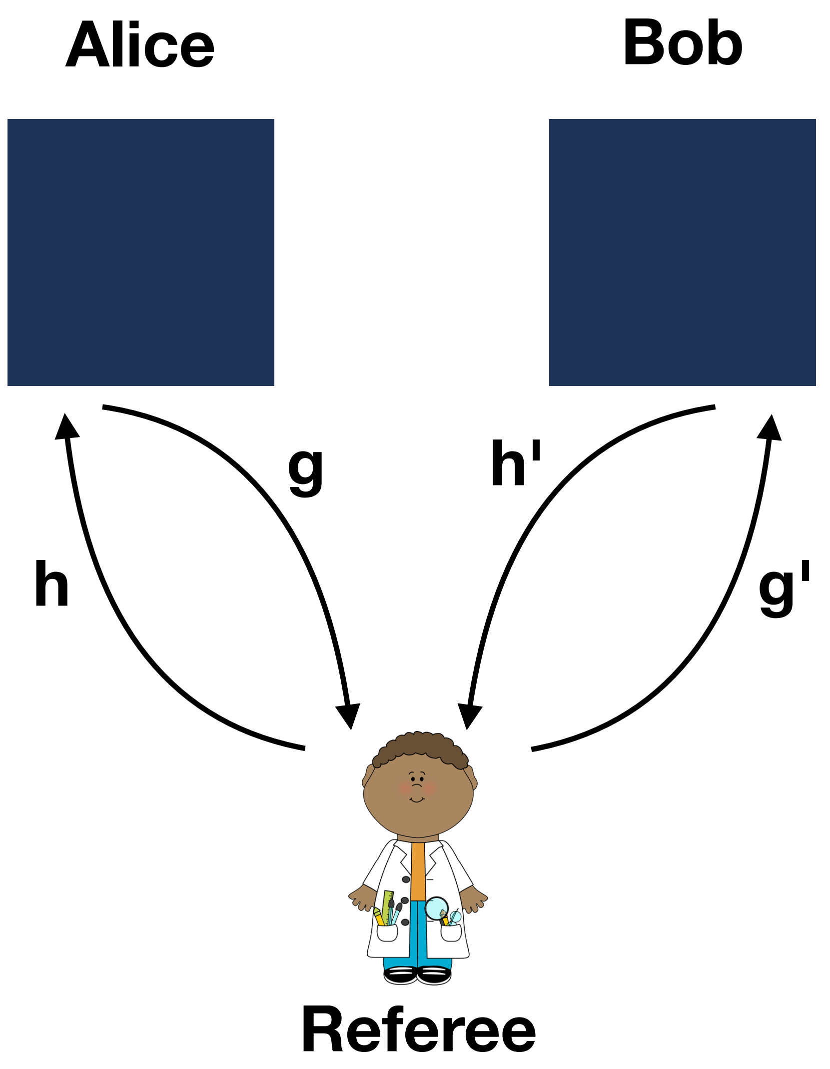


- Polynomially many messages and BPP verifier and entangled provers → MIP*.

Interactive Proof Complexity Classes



(G, H)-Isomorphism Game



Intuition: Alice and Bob want to convince referee that $G \cong H$

1. Referee sends vertices from either graph
2. Players respond with vertices from other graph
3. Win if vertex relationships preserved

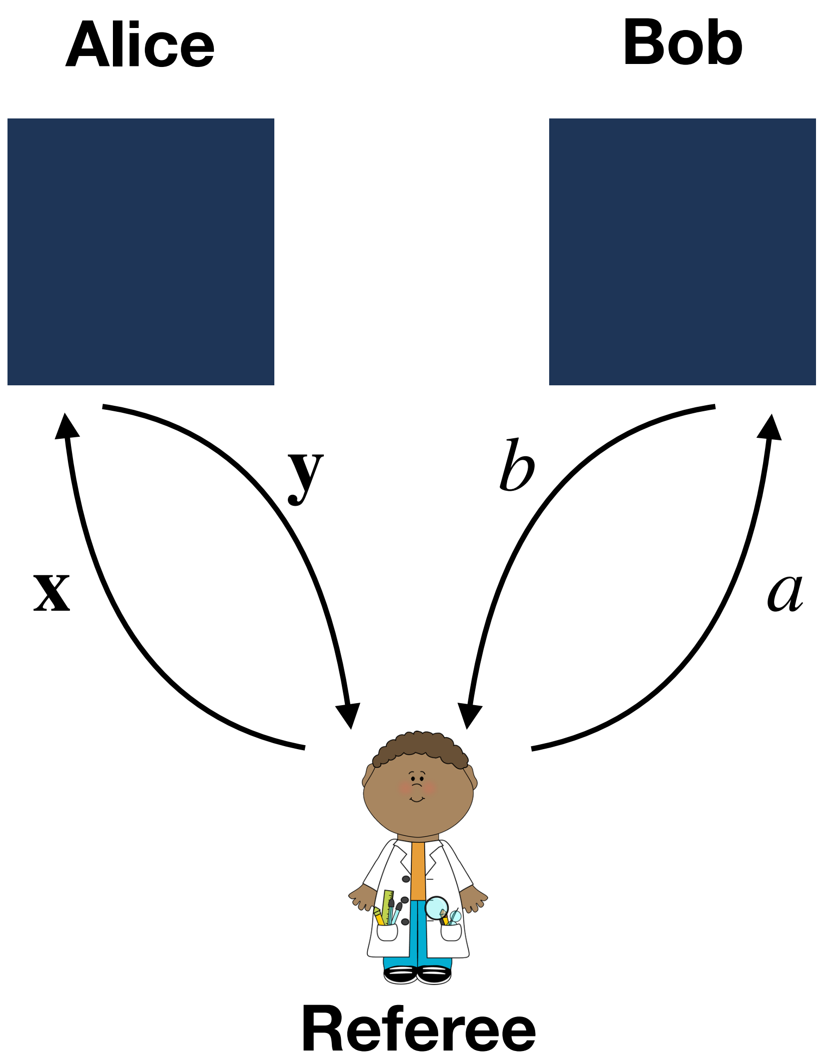
[1] Albert Atserias, Laura Mančinska, David E. Roberson, Robert Šámal, Simone Severini, Antonios Varvitsiotis "Quantum and non-signalling graph isomorphisms" *arXiv preprint arXiv:1611.09837* (2017).

(G, H)-Isomorphism Game

Strategy	Matrix Formulation	Homomorphism Count	Complexity
Classical	Permutation Matrix	All graphs	Quasi-Polynomial
Quantum	Magic Unitary	?	Undecidable
Commuting	Projective Permutation Matrix	Planar graphs	CoRE-complete
Non-signalling	Doubly Stochastic Matrix	Trees	Polynomial

[1] Albert Atserias, Laura Mančinska, David E. Roberson, Robert Šámal, Simone Severini, Antonios Varvitsiotis "Quantum and non-signalling graph isomorphisms" *arXiv preprint arXiv:1611.09837* (2017).

Homomorphism Game [1]



Intuition: Given finite structures $\mathscr{A}, \mathscr{B}$ Alice and Bob want to convince referee that $\mathscr{A} \to \mathscr{B}$

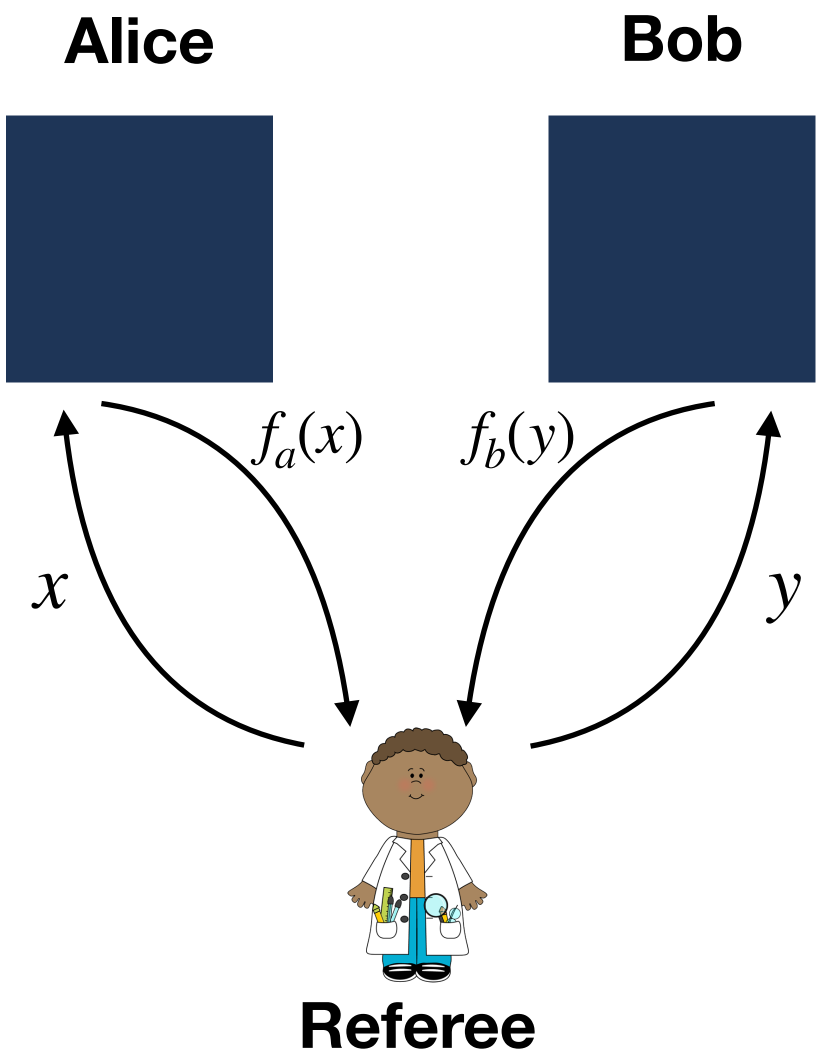
1. Referee sends Alice a tuple $\mathbf{x} \in R^{\mathscr{A}}$ and Bob an element $a \in A$

2. Alice responds with a tuple $\mathbf{y} \in \mathscr{B}^k$ and Bob responds with an element $b \in B$

3. Alice and Bob win if:
   A. $\mathbf{y} \in \mathscr{R}^b$
   B. $a = \mathbf{x}_i \implies b = \mathbf{y}_i$

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Classical Strategies

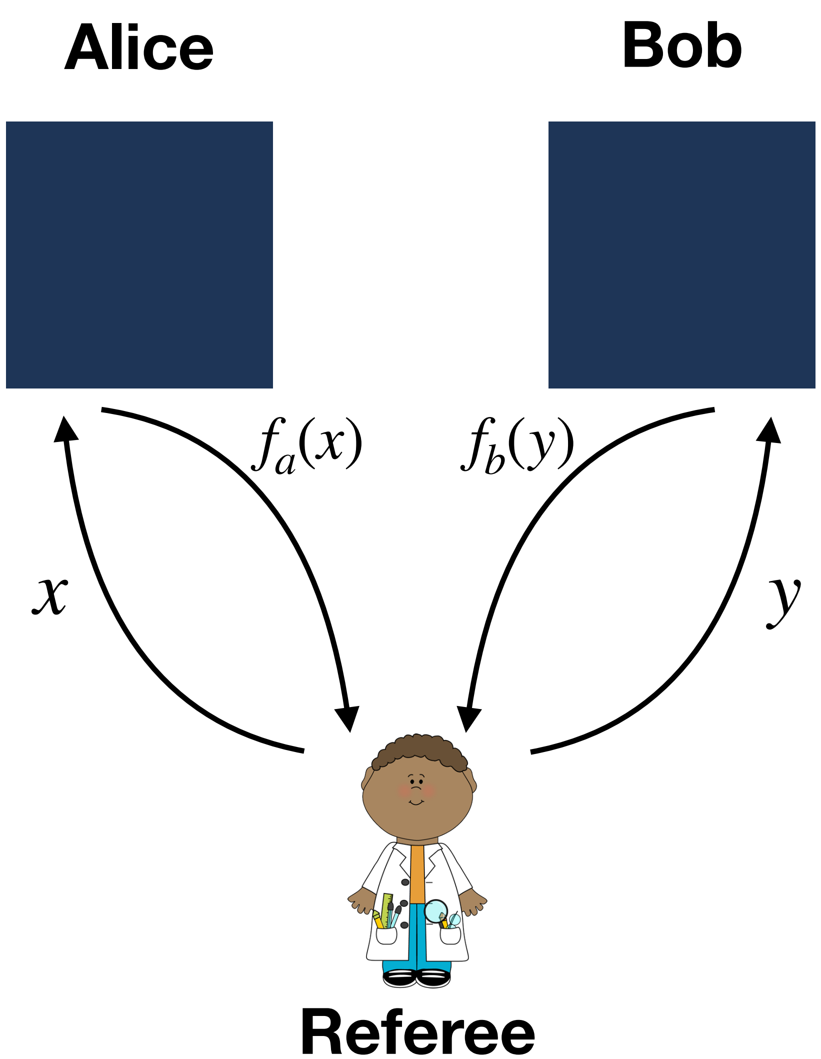


- Deterministic functions f_a and f_b.

$$p(f_a(x), f_b(y) \,|\, x, y) = 1$$

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Classical Strategies



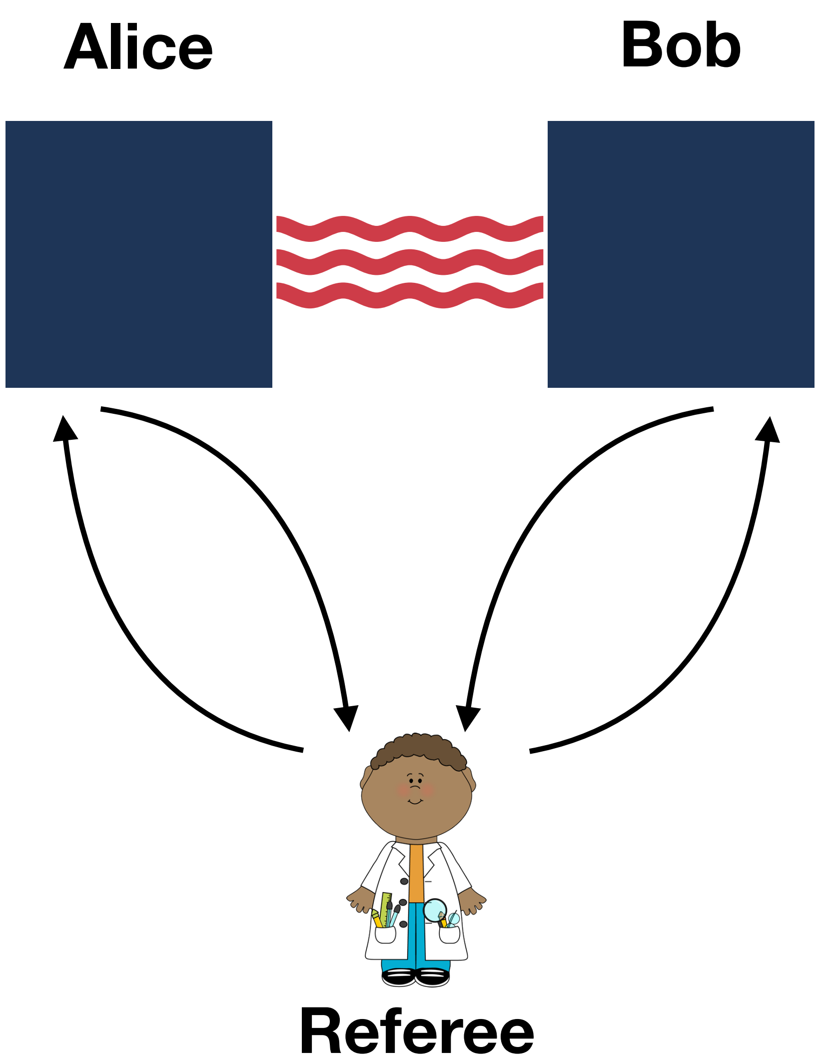
- Deterministic functions f_a and f_b.

$$p(f_a(x), f_b(y) \,|\, x, y) = 1$$

There exists a perfect classical strategy for the homomorphism game iff $\mathscr{A} \to \mathscr{B}$

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Quantum Strategies



- Alice and Bob to share an entangled state which they can perform measurements on.
- We say there is a quantum homomorphism $\mathscr{A} \xrightarrow{q} \mathscr{B}$ whenever a quantum perfect strategy exists for the homomorphism game.

Theorem [1]: $\mathscr{A} \xrightarrow{q} \mathscr{B}$ iff there exists a kleisli morphism $\mathscr{A} \to \mathbb{Q}_{=d}\mathscr{B}$ for some d.

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Monads and Comonads

A **monad** is a triple (M, η, μ) where:

1. $M\colon C \to C$ is an endofunctor.
2. The **unit** $\eta\colon id_C \to M$ is a natural transformation.
3. The **multiplication** $\mu\colon M^2 \to M$ is a natural transformation.

And the following equations hold:

$$\mu \circ M\mu = \mu \circ \mu M; \quad \mu \circ M\eta = \mu \circ \eta M = id_M$$

A **comonad** is a triple (W, ϵ, δ) where:

1. $W\colon C \to C$ is an endofunctor.
2. The **counit** $\eta\colon W \to id_C$ is a natural transformation.
3. The **compultiplication** $\delta\colon M \to M^2$ is a natural transformation.

And the following equations hold:

$$W\delta \circ \delta = \delta W \circ \delta; \quad W\epsilon \circ \delta = \epsilon W \circ \delta = id_W$$

Monads and Comonads

Monads model effectful computation e.g. nondeterminism, probabilities.

An effectful computation from A to B is represented as a morphism $A \to MB$ and can be composed in the **Kleisli** category $Kl(M)$ of a monad M:

- $Obj(Kl(M)) = Obj(C)$
- $Hom_{Kl(M)}(A, B) = Hom(A, MB)$
- $id_x = \eta_x$
- $g \circ f = X \xrightarrow{f} MY \xrightarrow{g^*} MZ$ where:
  - $f : X \to MY$ and $g : Y \to MZ$
  - $g^* = \mu_z \circ Mg$

Comonads model contextual computation e.g. list prefixes, tree nodes.

A contextual computation from A to B is represented as a morphisms $WA \to B$ and can be composed in the **coKleisli** category $coKl(W)$ of a comonad W:

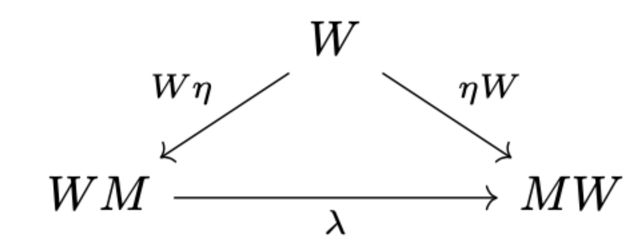
- $Obj(coKl(W)) = Obj(C)$
- $Hom_{coKl(W)}(A, B) = Hom(WA, B)$
- $id_x = \epsilon_x$
- $g \circ f = WX \xrightarrow{f^*} WY \xrightarrow{g} Z$ where:
  - $f : WX \to Y$ and $g : WY \to Z$
  - $f^* = Wf \circ \delta_x$

Natural question: When can computations that are both contextual, and effectful be modelled as morphisms in a suitable category?

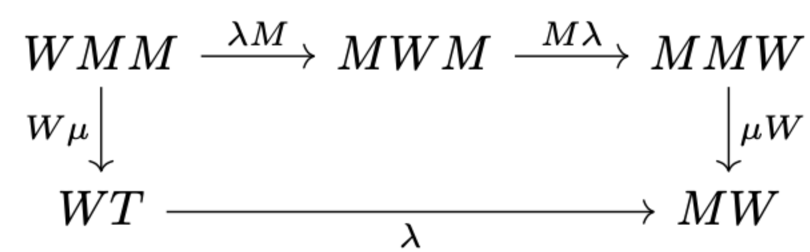
Distributive Laws

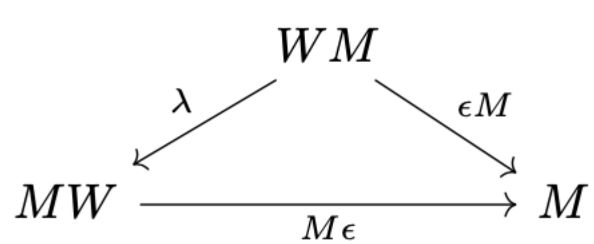
A (mixed) **distributive law** of a comonad (W, ϵ, δ) over a monad (M, η, μ) is a natural transformation $\lambda :\ W \circ M \Rightarrow M \circ W$ satisfying four axioms:

Unit

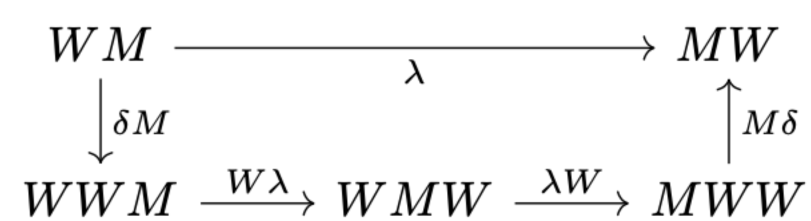


Multiplication





Counit



Comultiplication

Note that each axiom can be satisfied independently of the others. In particular, we say that there exists a **Kleisli law** between W and M whenever the unit and multiplication axioms are satisfied.

BiKleisli Categories

Natural question: When can computations that are both contextual, and effectful be modelled as morphisms in a suitable category?

A mixed distributive law $\lambda: W \circ M \Rightarrow M \circ W$ allows us to define a **biKleisli category** $biKl(W, M)$ whose morphisms are of the form $WA \to MB$:

- $Obj(biKl(W, M)) = Obj(C)$
- $Hom_{biKl(W,M)}(A, B) = Hom(WA, MB)$
- $id_x = \eta_x \circ \epsilon_x$
- $g \circ f = WX \xrightarrow{f^*} WMY \xrightarrow{\lambda_Y} MWY \xrightarrow{g^*} MZ$ where:

$biKl(W, M)$ can be seen as the Kleisli category of M lifted to $coKl(W)$, or equivalently as the coKleisli category of W lifted to $Kl(M)$.

Motivating Question

Is there a distributive law for game comonads G_k over Q_d?

If the answer is yes we can define a bikleisli category with morphisms of the form $G_k\mathscr{A} \to Q_d\mathscr{B}$. This would allow us to talk about quantum winning strategies for duplicator.

No-Go Theorems

- Distributive laws are not guaranteed to exist, and even when they do, finding them is often difficult.
- A result attributed to Plotkin shows that the powerset monad does not distribute over the distribution monad.
- [1] Vastly generalises this result to present several families of no-go-theorems for when the existence of distributive laws between pairs of monads is impossible.

Our contribution: First examples of no-go results for comonad-monad distributive laws.

[1] Zwart, Maaike, and Dan Marsden. "No-go theorems for distributive laws." In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1-13. IEEE, 2019.

Prefix-List and Power Set

- The non-empty powerset monad (P, η, μ) on $\mathbf{SET}$ is given by:
  1. $P(X)$ is the set of subsets of X.
  2. $\eta_X(x)$ is the singleton set $\{x\}$.
  3. μ_X takes a union of sets.

Also known as non-empty list. Isomorphic to suffix list.

- The prefix list comonad (N, ϵ, δ) on $\mathbf{SET}$ is given by:
  1. $N(X)$ is the set of all non-empty lists over X.
  2. $\epsilon_X[x_1, \ldots, x_n] = x_n$.
  3. $\delta_X[x_1, \ldots, x_n] = [[x_1], [x_1, x_2], \ldots, [x_1, x_2, \ldots x_n]]$.

Plotkin Style Counter-Example

Collapse, Swap and Tag Lemma: There is a unique pointed Kleisli law $\lambda^N \colon NP \to PN$ with components given by:

$$\lambda^N_X[X_1, \ldots, X_N] = \{[x_1, \ldots, x_n] \mid x_i \in X_i\}$$

$$[\{a,b\},\{c\}] \xrightarrow{\lambda_X} \{[a,c],[b,c]\} \xrightarrow{P\delta_X} \{[[a],[a,c]],[[b],[b,c]]\}$$

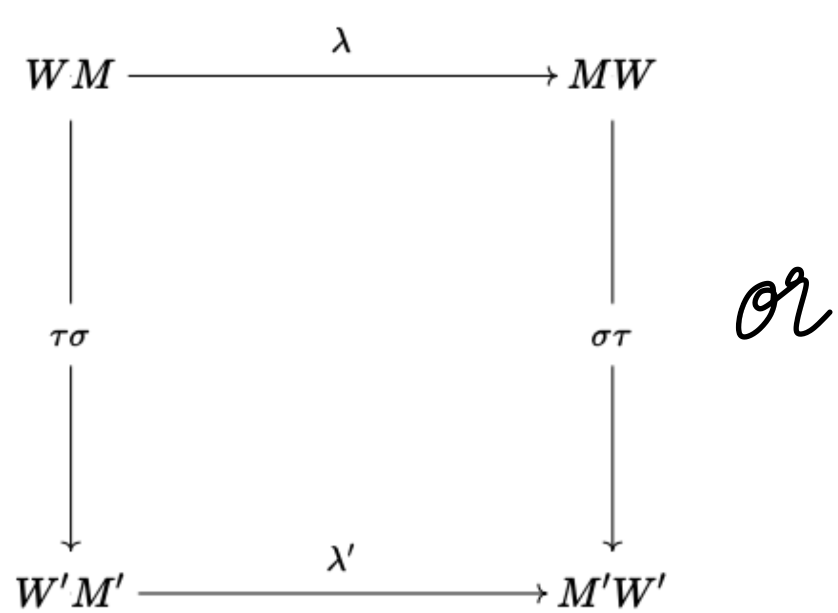
$$[\{a,b\},\{c\}] \xrightarrow{\delta_{PX}} [[\{a,b\}],[\{a,b\},\{c\}]] \xrightarrow{N\lambda_X} [\{[a],[b]\},\{[a,c],[b,c]\}] \xrightarrow{\lambda_{NX}} \{[[a],[a,c]],[[b],[a,c]],[[a],[b,c]],[[b],[b,c]]\},$$

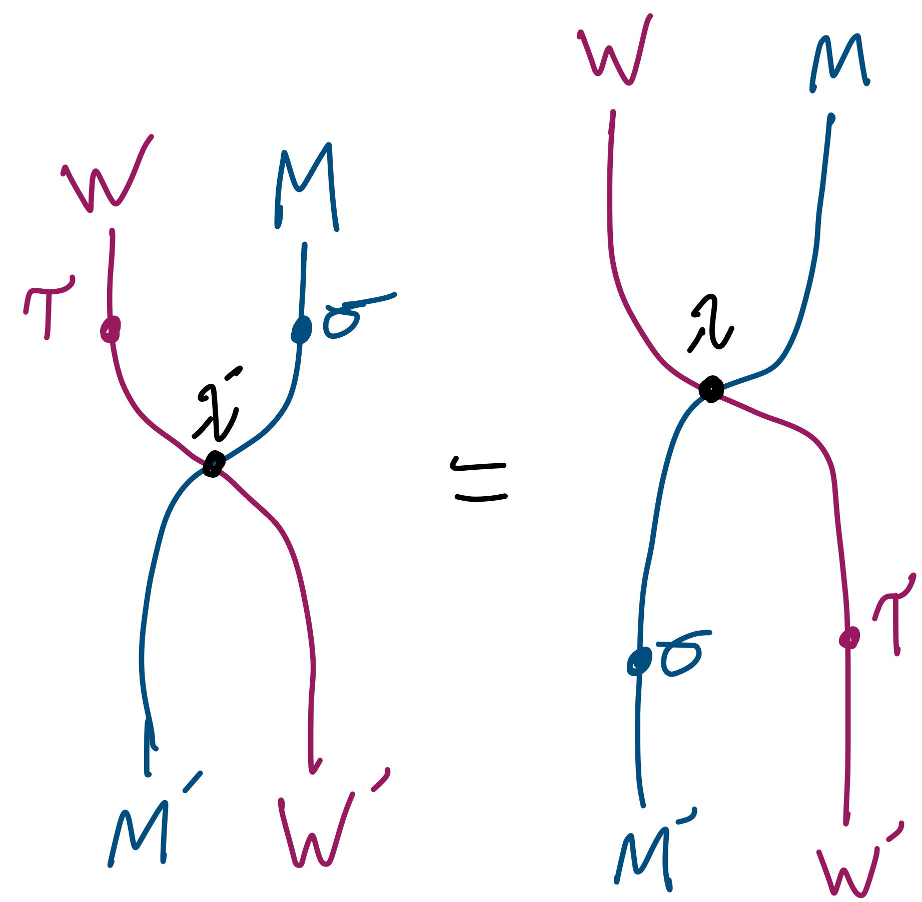
$$\neq$$

Theorem: There is no distributive law of the comonad (N, ϵ, δ) over the monad (P, η, μ)

Transfer Theorems

Theorem : Assume the following diagram commutes:





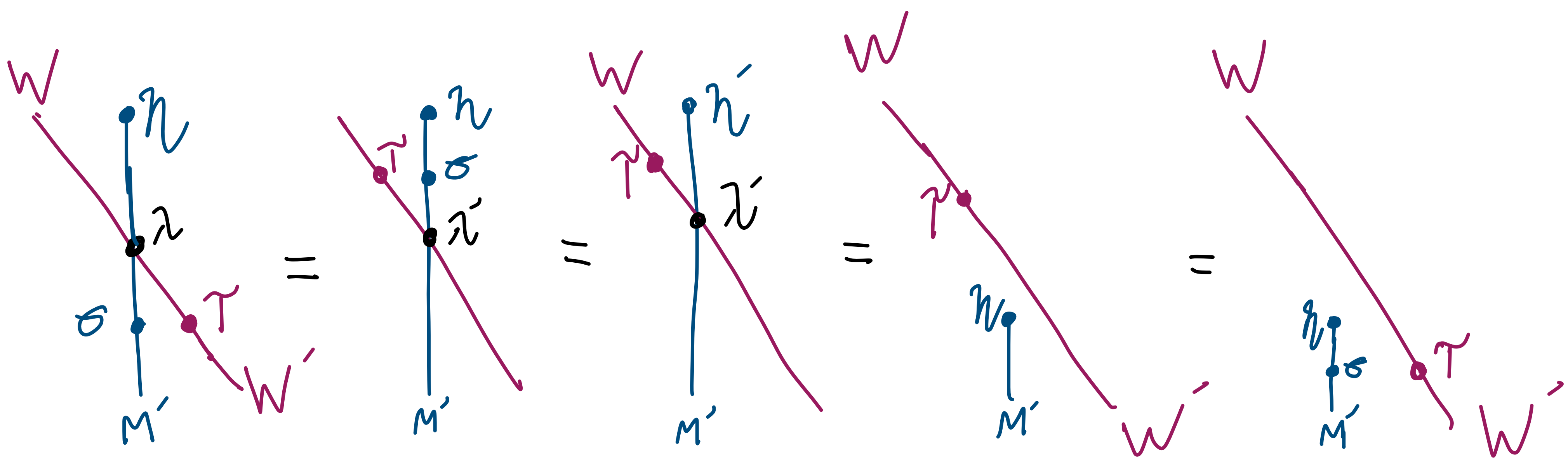
Then we have (with some caveats):

1. If λ' is a distributive law and τ, σ are monic, then λ is a distributive law.
2. If λ is a distributive law and τ, σ are epic, then λ' is a distributive law.

τ is a comonad map, that is: $\delta' \circ \tau = (\tau \star \tau) \circ \delta$ and $\epsilon' \circ \tau = \epsilon$
σ is a monad map, that is: $\rho \circ \mu = \mu' \circ (\rho \star \rho)$ and $\rho \circ \eta = \eta'$

Proof sketch

- Transfer theorems can be proven elegantly using string diagrams:



Proof Sketch

- Or alternatively using straightforward (but tedious) algebra:

$$
\begin{aligned}
(\sigma \star \tau) \circ \lambda' \circ W'\eta' &= \lambda \circ (\tau \star \sigma) \circ W'\eta' && (8)\\
&= \lambda \circ \tau M \circ W'\sigma \circ (W'\eta') && \text{definition of } \star\\
&= \lambda \circ \tau M \circ W'(\sigma \circ \eta') && F(\alpha \circ \beta) = F\alpha \circ F\beta\\
&= \lambda \circ \tau M \circ W'\eta && \text{definition of monad map}\\
&= \lambda \circ (\tau \star \eta) && \text{definition of } \star\\
&= \lambda \circ W\eta \circ \tau \mathbf{id} && \text{definition of } \star\\
&= \eta W \circ \mathbf{id}\tau && \lambda \text{ unit axiom}\\
&= \eta \star \tau && \text{definition of } \star\\
&= M\tau \circ \eta W' && \text{definition of } \star\\
&= M\tau \circ (\sigma \circ \eta')W' && \text{definition of monad map}\star\\
&= M\tau \circ \sigma W' \circ \eta' W' && (\alpha \circ \beta)F = \alpha F \circ \beta F\\
&= (\sigma \star \tau) \circ \eta' W' && \text{definition of}\star
\end{aligned}
$$

Containers

- Containers are endofunctors $F : \mathbf{SET} \to \mathbf{SET}$ which have an associated set of shapes S and a set of positions $P(s)$ for each shape $s \in S$ such that F is the S-indexed coproduct of the exponential functors $(\_)^{P(s)}$.

- Over $\mathbf{SET}$, containers are equivalent to polynomial functors.

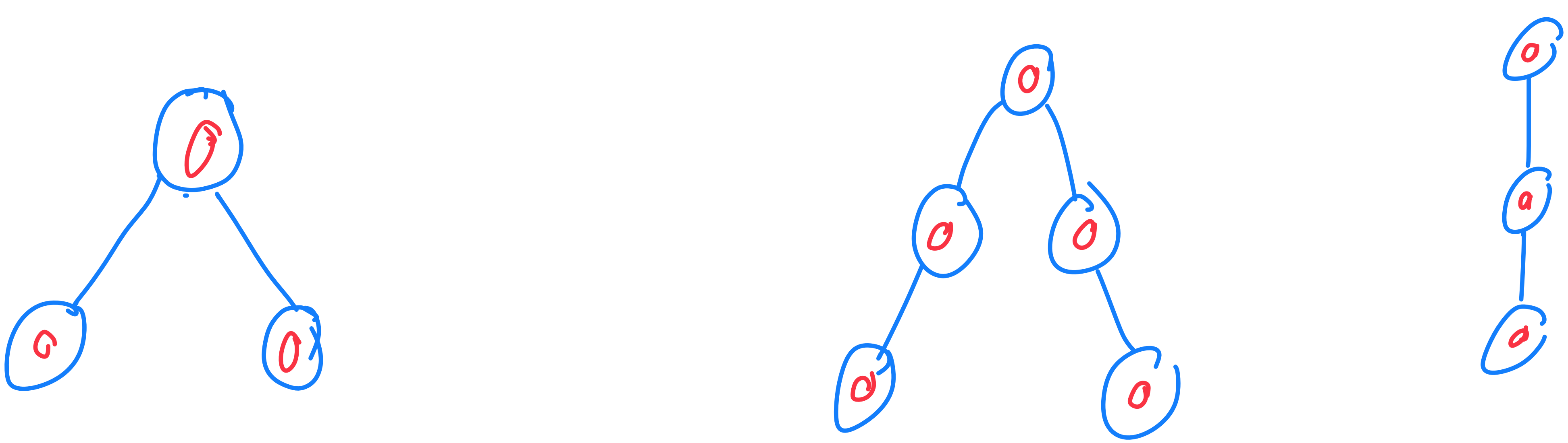
- Using the transfer theorems we will show how our results extend to many comonads whose underlying functor is a container (also known as directed containers).

[1] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Categories of containers. In International Conference on Foundations of Software Science and Computation Structures, pages 23–38. Springer, 2003.

Containers

- The prefix list functor N is a container where $S = Nat$ and $P(s) = \{1, 2, \ldots, s\}$

- The labelled binary suffix tree comonad $(B, \epsilon^B, \delta^B)$ on $\mathbf{SET}$ is given by:
    1. $B(X)$ is the set of all binary trees with nodes labelled by elements of X.
    2. $\epsilon_X(t)$ returns the root node of t.
    3. $\delta_X(t)$ replaces each node of t with the subtree rooted at that node.



Second No-Go Result

Lemma: There exists a monic comonad map $\tau^B : N \Rightarrow B$

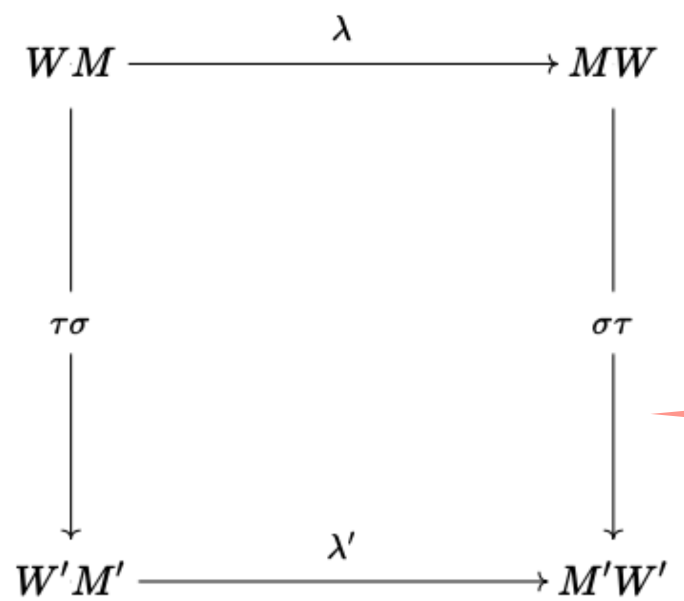
- Intuitively, τ^B_X sends a list $\mathbf{L}$ to a tree with a single path, which is the reverse of $\mathbf{L}$. It is easy to see that this map is monic.

Collapse, Swap and Tag Lemma: There is a unique kleisli law $\lambda^B : BP \Rightarrow PB$ with components:

$$\lambda^B_X(T) = \{t \,|\, \text{root}(t) \in \text{root}(T) \text{ and } x_1 \overset{t}{\rightsquigarrow} x_2 \implies x_1 \in X_1, x_2 \in X_2 s.t. X_1 \overset{T}{\rightsquigarrow} X_2\}$$

Proof Sketch

Theorem : Assume the following diagram commutes:



Set $M = M' = P$, $W = N$, $W' = B$, $\lambda = \lambda^N$, $\lambda' = \lambda^B$, $\tau = \tau^B$ and $\sigma = id^P$.

The conditions of this theorem are now satisfied thanks to the two lemmas.

The uniqueness of λ^B completes the proof.

Then we have:

- If λ' is a distributive law and τ, σ are monic, then λ is a distributive law.

Corollary: If there is a distributive law of $(B, \epsilon^B, \delta^B)$ over (P, η, μ) then there is also a distributive law of (N, ϵ, δ) over (P, η, μ).

More containers

- The underlined list comonad $(N^*, \epsilon^*, \delta^*)$ on $\mathbf{SET}$ is given by:
  1. $N^*(X)$ is the set of all pointed lists over X. A pointed list is a tuple $(\mathbf{L}, i)$ where $\mathbf{L}$ is a list and i refers to an index of $\mathbf{L}$.
  2. $\epsilon^*_X([x_1, \ldots, x_n], i) = x_i$.
  3. $\delta_X(\mathbf{L}, i) = ([(\mathbf{L},1), (\mathbf{L},2), \ldots, (\mathbf{L}, n)], i)$.

- Given k pebbles, the pebble list comonad $(N^k, \epsilon^k, \delta^k)$ on $\mathbf{SET}$ is given by:
  1. $N^k(X)$ is the set of non-empty list of *moves* (p, x) where $p \in [k], x \in X$.
  2. $\epsilon^*_X[(p_1, x_1), \ldots, (p_n, x_n)] = x_n$.
  3. $\delta_X[(p_1, x_1), \ldots, (p_n, x_n)] = [(p_1, \mathbf{L_1}), \ldots, (p_n, \mathbf{L}_n)]$ where $\mathbf{L_i} = [(p_1, x_1), \ldots, (p_i, x_i)]$

Generalised Theorem: Any Container has a pointed kleisli law over the powerset monad.

Sufficient Conditions

Theorem: Let (T, ϵ, δ) be a comonad. If T is a container then there exists a unique pointed kleisli law $\lambda^T : TP \to PT$.
If either of the following conditions are satisfied, there is no distributive law of (T, ϵ, δ) over (P, η, μ):

1. There exists a monic comonad morphism $\tau^T : N_2 \Rightarrow T$ such that $\lambda^T, \lambda^N, \tau^T, id^P$ satisfy the conditions of transfer theorem (1).
2. There exists an epic comonad morphism $\tau^T : T \Rightarrow N_2$ such that $\lambda^T, \lambda^N, \tau^T, id^P$ satisfy the conditions of transfer theorem (2).

Example: There is no distributive law of $(N^*, \epsilon^*, \delta^*)$ over (P, η, μ)

Example: There is no distributive law of $(N^k, \epsilon^k, \delta^k)$ over (P, η, μ)

Distribution Monads

- A commutative semiring is given by $\mathbb{S} = (S, 0_{\mathbb{S}}, 1_{\mathbb{S}}, +, .)$ such that $(S, 0_{\mathbb{S}} +, .)$ is an additive commutative monoid and $(S, 1_{\mathbb{S}}, .)$ is a multiplicative commutative monoid with multiplication distributing over addition.

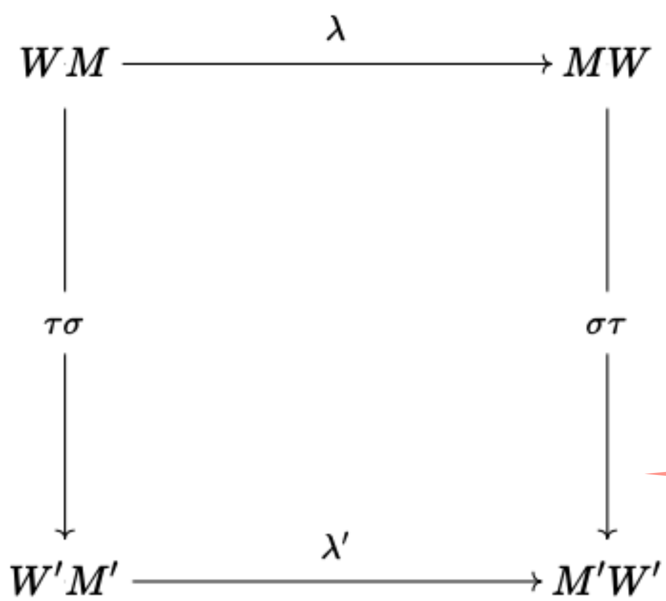
- The distribution monad for $\mathbb{S}$, $(\mathscr{M}_{\mathbb{S}}, \eta^D, \mu^D)$ is a monad on $\mathbf{SET}$ given by:
  1. $\mathscr{M}_{\mathbb{S}}(X) = \{\varphi : X \to S \mid \text{supp}(\varphi) \text{ is finite}\}$ s.t $\sum_i s_i = 1$.
  2. $\eta_X(x) = 1_{\mathbb{S}}x$
  3. $\mu_X(\sum_i s_i \varphi_i)(x) = \sum_i s_i \,.\, \varphi_i(x)$

- The standard probability distribution monad is recovered as the distribution monad for $\mathbb{S} = (\mathbb{R}_{\geq 0}, 0, 1, +, \times)$.

- The finite non-empty powerset monad is recovered as the distribution monad for the boolean semiring $\mathbb{B} = (\{0,1\}, 0, 1, \vee, \wedge)$.

Uniqueness up to support: If T is a container and $\lambda, \lambda' : T\mathscr{D}_{\mathbb{S}} \rightarrow \mathscr{D}_{\mathbb{S}}T$ are pointed kleisli laws then we have $supp(\lambda_X(t)) = supp(\lambda'_X(t))$.

Theorem : Assume the following diagram commutes:



Set $M = \mathscr{D}_{\mathbb{S}}, M' = P, W = W' = N,$
$\lambda = \lambda^{\mathscr{M}_{\mathbb{S}}}, \lambda' = \lambda^N, \tau = id^N$ and $\sigma = supp$.

The conditions of this theorem are now satisfied.

Uniqueness up to support completes the proof.

Then we have:

2. If λ is a distributive law and τ, σ are epic, then λ' is a distributive law.

Corollary: None of the comonads discussed distribute over $(\mathscr{D}_{\mathbb{S}}, \eta, \mu)$.
(Some cardinality caveats swept under the rug)

Quantum Monad

- The graded quantum monad $(\mathbb{Q}_=, \eta^q, \mu^{d,d'})$ is a graded monad on $\mathbf{R}(\sigma)$ given by:
  - $\mathbb{Q}_{=n}(\mathscr{X})$ is the set of all functions $p : X \to Proj(d)$ satisfiying $\sum_{x \in X} p(x) = I$.
  - $R^{\mathbb{Q}_{=d}\mathscr{X}}$ is the set of all tuples $(p_1, \ldots, p_k)$ satisfying:
    1. $\forall i, j \in [k], x, x' \in X : [p_i(x), p_j(x')] = \mathbf{0}$:
    2. $\forall (x_1, \ldots, x_k) \in X^k, if \mathbf{x} \notin R^{\mathscr{X}}$, then $p_1(x_1) \ldots p_k(x_k) = \mathbf{0}$
  - The natural transformations $\mathbb{Q}_{n=n'}$ are the identity.
  - $\eta^q_{\mathscr{X}}(x) = \delta_x$ where $\delta_x(x) = I_1, \delta_x(x') = \mathbf{0}$ if $x \neq x'$.
  - $\mu^{d,d'}_{\mathscr{X}}$ is given by tensor multiplication i.e. $\mu^{d,d'}_{\mathscr{X}}(P)(x) := \sum_{p \in \mathbb{Q}_{d'}} P(p) \otimes p(x)$.

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Quantum Monad

- The graded quantum monad $(\mathbb{Q}_=, \eta^q, \mu^{d,d'})$ is a graded monad on $\mathbf{R}(\sigma)$ given by:
  - $\mathbb{Q}_{=n}(\mathcal{X})$ is the set of all functions $p : X \to Proj(d)$ satisfiying $\sum_{x \in X} p(x) = I$.
  - ~~$R^{\mathbb{Q}_{=d}\mathcal{X}}$ is the set of all tuples $(p_1, \ldots, p_k)$ satisfying:~~
    1. ~~$\forall i, j \in [k], x, x' \in X : [p_i(x), p_j(x')] = \mathbf{0}$;~~
    2. ~~$\forall (x_1, \ldots, x_k) \in X^k, if \mathbf{x} \notin R^{\mathcal{X}}$, then $p_1(x_1) \ldots p_k(x_k) = \mathbf{0}$~~
  - The natural transformations $\mathbb{Q}_{n=n'}$ are the identity.
  - $\eta^q_{\mathcal{X}}(x) = \delta_x$ where $\delta_x(x) = I_1, \delta_x(x') = \mathbf{0}$ if $x \neq x'$.
  - $\mu^{d,d'}_{\mathcal{X}}$ is given by tensor multiplication i.e. $\mu^{d,d'}_{\mathcal{X}}(P)(x) := \sum_{p \in \mathbb{Q}_{d'}} P(p) \otimes p(x)$.

[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Quantum Monad

- If we only consider the action of $\mathbb{Q}_{=d}$ on the underlying universe of structures, we get a graded monad $(Q_{=}, \eta^q, \mu^{d,d'})$ on the category $\mathbf{SET}$.

- We can “Degrade”[1] this graded monad to obtain a normal monad (Q, η^q, μ^q):
  1. $Q(X)$ is the set of all functions $p : X \to Proj$ satisfying $\sum_{x \in X} p(x) = I$
  2. $\eta^q_X(x) = \delta_x$ where $\delta_x(x) = I_1, \delta_x(x') = \mathbf{0}$ if $x \neq x'$.
  3. $\mu^q_{\mathcal{X}}$ is given by tensor multiplication i.e. $\mu^{d,d'}_{\mathcal{X}}(P)(x) := \sum_{p \in Q} P(p) \otimes p(x)$.

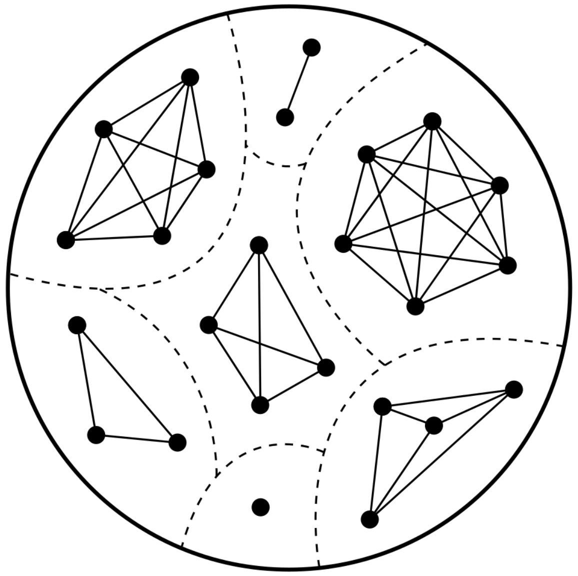
- This can be seen as a variant of the distribution monad where probability distributions are replaced with PVMs.

[1] Dylan McDermott, Maciej Piróg, and Tarmo Uustalu. 2020. Degrading Lists

Kleisli Categories and Quantum Homomorphisms

Theorem [1]: $\mathscr{A} \xrightarrow{q} \mathscr{B}$ iff there exists a kleisli morphism $\mathscr{A} \to \mathbb{Q}_d\mathscr{B}$ for some d.

Theorem: $\mathscr{A} \xrightarrow{q} \mathscr{B}$ iff there exists a kleisli morphism $\mathscr{A} \to Q\mathscr{B}$.



[1] Abramsky, Samson, Rui Soares Barbosa, Nadish De Silva, and Octavio Zapata. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

Quantum Monad

Theorem: There is no comonad-monad distributive law of the form $WQ \to QW$ for any comonad W seen thus far.

Open problem: Rule out existence of graded distributive laws of the form $W_{=a}Q_{=b} \Rightarrow Q_{=i}W_{=j}$

Monad-Comonad Distributive Laws

A Go Theorem

Theorem: There exists a distributive law $\lambda : D_{\mathbb{S}}N \Rightarrow ND_{\mathbb{S}}$ of the prefix list comonad over the distribution monad for $\mathbb{S}$ with components given by:

$$\lambda_X(s_1L_1 + \ldots + s_NL_N) = [s_1L_1[-k] + \ldots + s_NL_N[-k], s_1L_1[-k+1] + \ldots + s_NL_N[-k+1], s_1L_1[-1] + \ldots + s_NL_N[-1]]$$

Where $k = min(length(L_i))$ and $L[-i]$ refers to the ith last element of L.

Open problem: Is λ a distributive law of $\mathbb{Q}$ over $\mathbb{E}$?

Comonad Generalisation

- When we view the prefix list functor as a container, we can think of the distributive law described previously as a two-step process:

  1. Identifying the common subshape of all the non-empty lists in $D_{\mathbb{S}}N(X)$.
  2. Merge together the elements at each position of the common subshape, while ignoring elements at other positions.

- This idea can be adapted and used to come up with distributive laws for other containers. For example, we can construct distributive laws of $D_{\mathbb{S}}$ over the stream comonad or the binary suffix tree comonad.

Open problem: How abstractly can this distributive law be stated? Does it work for all comonads whose underlying functor is a container?

Open problem: More generally, can we come up with an axiomatic account of when mixed distributive laws can or cannot exist?