Comonadic Semantics for Hybrid Logic

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Outline

- Background on hybrid logic.
- The hybrid game comonad.
- Local character and semantic characterisation.

Hybrid Temporal Logic

Syntax of HTL

$$\varphi ::= \underbrace{p \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi'}_{\textbf{propositional}} \mid \overbrace{\Box\varphi \mid \Diamond\varphi \mid \Box^{-}\varphi \mid \Diamond^{-}\varphi}^{\textbf{2-way modal}} \mid \underbrace{x \mid {\downarrow}x.\,\varphi \mid @_x\varphi}_{\textbf{hybrid}}$$

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The *positive fragment* excludes $\Box$, $\Box^{-}$ and $\neg$.

Hybrid Temporal Logic

Semantics

Models are relational structures over a signature with unary relation symbols and a single binary relation symbol E.

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Propositions and Boolean connectives:

$$\begin{aligned}
\mathsf{ST}_x(p) &= P(x) \\
\mathsf{ST}_x(\neg\varphi) &= \neg\,\mathsf{ST}_x(\varphi) \\
\mathsf{ST}_x(\varphi \wedge \varphi') &= \mathsf{ST}_x(\varphi) \wedge \mathsf{ST}_x(\varphi') \\
\mathsf{ST}_x(\varphi \vee \varphi') &= \mathsf{ST}_x(\varphi) \vee \mathsf{ST}_x(\varphi')
\end{aligned}$$

Hybrid Temporal Logic

Semantics

Models are relational structures over a signature with unary relation symbols and a single binary relation symbol E.
Modalities:

$$\begin{aligned}
\mathsf{ST}_x(\Box\varphi) &= \forall y.[E(x,y) \rightarrow \mathsf{ST}_y(\varphi)] \\
\mathsf{ST}_x(\Diamond\varphi) &= \exists y.[E(x,y) \wedge \mathsf{ST}_y(\varphi)] \\
\mathsf{ST}_x(\Box^-\varphi) &= \forall y.[E(y,x) \rightarrow \mathsf{ST}_y(\varphi)] \\
\mathsf{ST}_x(\Diamond^-\varphi) &= \exists y.[E(y,x) \wedge \mathsf{ST}_y(\varphi)]
\end{aligned}$$

Hybrid Temporal Logic

Semantics

Models are relational structures over a signature with unary relation symbols and a single binary relation symbol E.

Hybrid connectives:

$$\mathsf{ST}_x(\downarrow x'.\varphi) = \mathsf{ST}_x(\varphi)[x/x']$$
$$\mathsf{ST}_x(@_{x'}\varphi) = \mathsf{ST}_x(\varphi)[x'/x]$$
$$\mathsf{ST}_x(x') \; = \; x = x'$$

Example Hybrid Formulae

Example

1. $\mathsf{ST}_x(\downarrow x'.\Diamond x')$

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$$\exists y.E(x,y) \wedge x = y$$

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$$E(x, x)$$

Example Hybrid Formulae

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$$E(x, x)$$

2. $\mathsf{ST}_x(@_{x'} p)$

Example Hybrid Formulae

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1. $\mathsf{ST}_x(\downarrow x'.\Diamond x')$

$$E(x, x)$$

2. $\mathsf{ST}_x(@_{x'} p)$

$$P(x)[x'/x]$$

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3. $\mathsf{ST}_x(@_{x'} \Diamond y')$

Example Hybrid Formulae

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$$E(x,x)$$

2. $\mathsf{ST}_x(@_{x'} p)$

$$P(x')$$

3. $\mathsf{ST}_x(@_{x'}\Diamond y')$

$$\exists x_1.E(x',x_1) \wedge x_1 = y'$$

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4. $\mathsf{ST}_x(@_{x'} y')$

$$x' = y'$$

The Hybrid Game

The **Hybrid game** between pointed structure $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ has initial position is $a_0 = a$, $b_0 = b$. In each round i, Spoiler moves by either

- choosing an $a_j \in A$ such that for some $i < j$, $E^{\mathfrak{A}}(a_i, a_j)$ or $E^{\mathfrak{A}}(a_j, a_i)$, to which Duplicator responds by choosing a $b_j \in B$; or
- choosing a $b_j \in B$ such that for some $i < j$, $E^{\mathfrak{B}}(b_i, b_j)$ or $E^{\mathfrak{B}}(b_j, b_i)$, to which Duplicator responds by choosing an $a_j \in A$.

The winning condition for Duplicator is that the correspondence $a_i \mapsto b_i$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

The Game / Logic Correspondence

- Duplicator has a winning strategy for the one-sided k-round game iff every positive Hybrid formula φ of depth $\leq k$:

$$(\mathfrak{A}, a) \models \varphi \quad \Rightarrow \quad (\mathfrak{B}, b) \models \varphi$$

- Duplicator has a winning strategy for the two-sided k-round game iff for every Hybrid formula φ of depth $\leq k$:

$$(\mathfrak{A}, a) \models \varphi \quad \Leftrightarrow \quad (\mathfrak{B}, b) \models \varphi$$

Hybrid Game Comonad

Game Comonads

Game comonads give a semantic characterisation of model equivalence games[1][2]. For a given logic $\mathcal{L}$, a game comonad is typically an indexed comonad $\mathbb{C}_k$ on relational structures such that:

- Morphisms $\mathbb{C}_k(\mathfrak{A}) \to \mathfrak{B}$ correspond to winning strategies for Duplicator in the one sided model comparison game, up to resource depth k.
- The existence of a span of open pathwise embeddings $F_k(\mathfrak{A}) \leftarrow R \to F_k(\mathfrak{B})$ in the Eilenberg-Moore category corresponds to a winning strategy for Duplicator in the full model comparison game.

[1] S. Abramsky, A. Dawar, and P. Wang. "The pebbling comonad in finite model theory". In: *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. IEEE, 2017, pp. 1–12.

[2] S. Abramsky and N. Shah. "Relating structure and power: Comonadic semantics for computational resources". In: *Journal of Logic and Computation* 31.6 (2021), pp. 1390–1428.

Hybrid Game Comonad

A Hybrid Comonad

To capture Hybrid logic, we work with pointed structures. For structure $(\mathfrak{A}, a)$, we define a structure $\mathbb{H}_k(\mathfrak{A}, a)$ with:

Universe: Non-empty sequences $\langle a_0, a_1, \ldots, a_l \rangle$ such that $a_0 = a$, and for all j with $0 < j \leq l$, for some i, $0 \leq i < j$, $E^{\mathfrak{A}}(a_i, a_j)$ or $E^{\mathfrak{A}}(a_j, a_i)$.

Counit: $\varepsilon_{\mathfrak{A}}(\langle a_0, \ldots a_l \rangle) = a_l$.

Relations: $R^{\mathbb{E}_k \mathfrak{A}}(s_1, \ldots, s_n)$ holds iff the s_i are pairwise comparable and $R^{\mathfrak{A}}(\varepsilon_{\mathfrak{A}}(s_1), \ldots, \varepsilon_{\mathfrak{A}}(s_n))$.

Point: $\langle a \rangle$.

Coextension: Given a morphism $h : \mathbb{H}_k(\mathfrak{A}, a) \to (\mathfrak{B}, b)$, we define $h^* : \mathbb{H}_k(\mathfrak{A}, a) \to \mathbb{H}_k(\mathfrak{B}, b)$ by

$$h^*(\langle a, a_1, \ldots, a_i \rangle) = \langle h(\langle a \rangle), \ldots, h(\langle a, a_1, \ldots, a_i \rangle) \rangle$$

Hybrid Game Comonad

Equality and I-relations

Given a vocabulary σ, define:

$$\sigma^+ = \sigma \cup \{I\}$$

There is a full and faithful embedding
$J : \mathsf{Struct}_\star(\sigma) \to \mathsf{Struct}_\star(\sigma^+)$ such that $I^{J(\mathfrak{A},a)}$ is the identity on A.

Hybrid Game Comonad

Game Comonad Results

- The triple $(\mathbb{H}_k, \varepsilon, (-)^*)$ is a comonad in Kleisli form.
- The composite $\mathbb{H}_k^+ = \mathbb{H}_k^I \circ J$ is a relative comonad.
- Kleisli morphisms have the form $\mathbb{H}_k^I J(\mathfrak{A}, a) \to J(\mathfrak{B}, b)$, and bijectively correspond to winning strategies for Duplicator in the one-sided game from $(\mathfrak{A}, a)$ to $(\mathfrak{B}, b)$.
- There is a span of open pathwise embeddings $F_k(\mathfrak{A}, a) \leftarrow R \to F_k(\mathfrak{B}, b)$ iff Duplicator has a winning strategy in the two-sided hybrid game between those structures.

Semantic Characterisation

Gaifman Graph Distance

Recall the *Gaifman graph* of a σ-structure $\mathfrak{A}$ has:

Vertices : The universe of $\mathfrak{A}$.

Edges : Pairs of distinct a, a' appearing in some $\overline{a}$ such that there exists $R \in \sigma$ with $R^{\mathfrak{A}}(\overline{a})$.

Yields a metric on $\mathfrak{A}$ via minimum path length.

Semantic Characterisation

Definitions

- Define $\mathbb{S}_k(\mathfrak{A}, a)$ to be $(\mathfrak{A}[a;k], a)$, where $\mathfrak{A}[a;k]$ is the substructure of $\mathfrak{A}$ induced by the closed ball $A[a;k]$ induced by the Gaifman graph distance in $\mathfrak{A}$.
- Define $\mathbb{S}(\mathfrak{A}, a) := \bigcup_{k \in \mathbb{N}} \mathbb{S}_k(\mathfrak{A}, a)$.
- Say that a first-order formula $\varphi(x)$ is *invariant under generated substructures* if for all $(\mathfrak{A}, a)$ in $\mathrm{Struct}_\star(\sigma)$: $(\mathfrak{A}, a) \models \varphi \iff \mathbb{S}(\mathfrak{A}, a) \models \varphi$.
- Say that a first-order formula $\varphi(x)$ is *invariant under k-generated substructures* if for all $(\mathfrak{A}, a)$ in $\mathrm{Struct}_\star(\sigma)$: $(\mathfrak{A}, a) \models \varphi \iff \mathbb{S}_k(\mathfrak{A}, a) \models \varphi$.
- We say that a sentence φ is *invariant under disjoint extensions* if for all $(\mathfrak{A}, a)$, $\mathfrak{B}$:

$$(\mathfrak{A}, a) \models \varphi \iff (\mathfrak{A} + \mathfrak{B}, a) \models \varphi$$

Semantic Characterisation

Theorem (Characterisation Theorem)

For any first-order formula $\varphi(x)$ with quantifier rank q, the following are equivalent:

1. *φ is invariant under generated substructures.*
2. *φ is invariant under $q2^q$-generated substructures.*
3. *φ is invariant under disjoint extensions.*
4. *φ is equivalent to a sentence ψ of hybrid temporal logic with modal depth $\leq q2^q$.*

Proof Idea One

Hybrid to FO Strategies via Resources

Lemma

For all $k, q > 0$,

$$\textit{if} \quad \mathbb{S}_k(\mathfrak{A}, a) \equiv_{kq}^{\mathsf{HTL}} \mathbb{S}_k(\mathfrak{B}, b) \quad \textit{then} \quad \mathbb{S}_k(\mathfrak{A}, a) \equiv_q \mathbb{S}_k(\mathfrak{B}, b)$$

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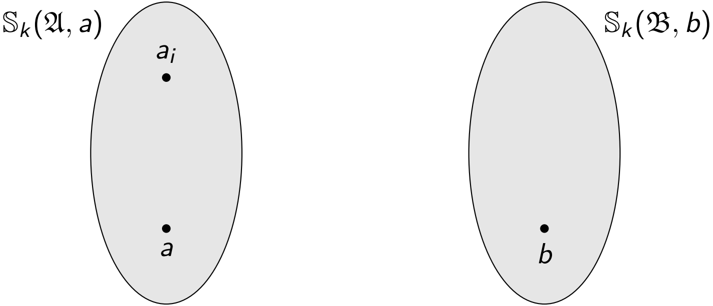

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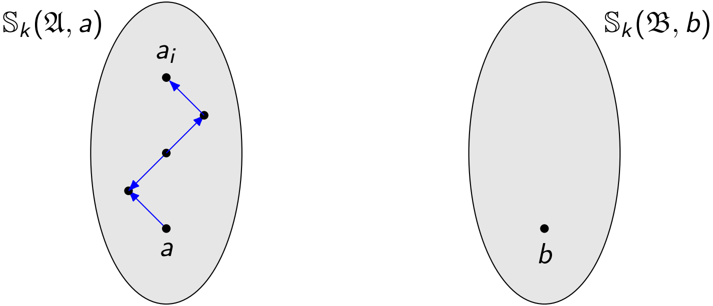

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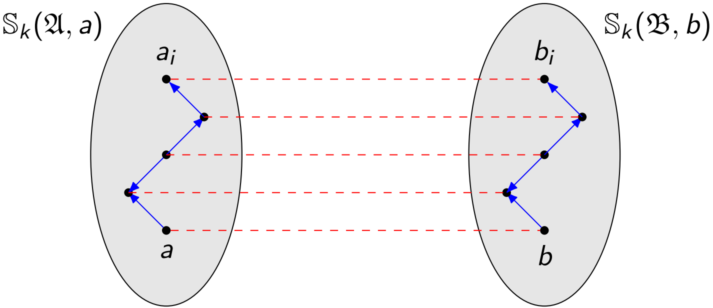


Proof Idea Two

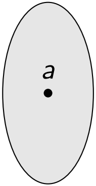
The Workspace Lemma

Lemma (Workspace Lemma)

Given $(\mathfrak{A}, a)$ *and* $q > 0$*, there is a structure* $\mathfrak{B}$ *such that*

$$(\mathfrak{A} + \mathfrak{B}, a) \equiv_q (\mathfrak{A}[a; k] + \mathfrak{B}, a),$$

where $k = 2^q$*. Moreover,* $|B| \leq 2q|A|$*. Hence if* $\mathfrak{A}$ *is finite, so is* $\mathfrak{B}$*.*



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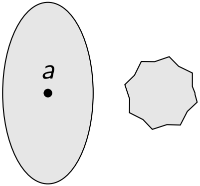

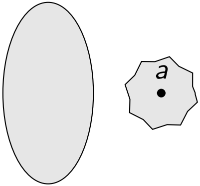

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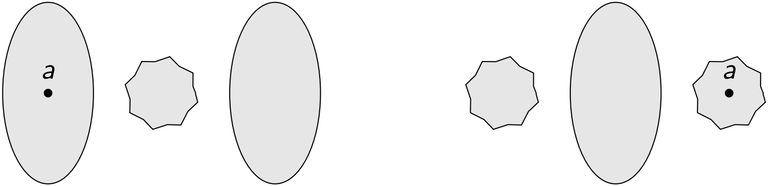

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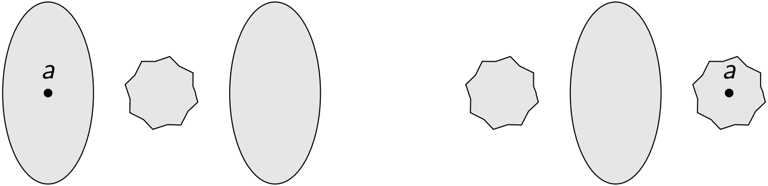

$$\mathfrak{B} = \underbrace{\mathfrak{A} + \ldots + \mathfrak{A}}_{q \textbf{ copies}} + \underbrace{\mathfrak{A}[a; k] + \ldots + \mathfrak{A}[a; k]}_{q \textbf{ copies}}$$

Final Remarks

- Everything in the paper can be extended to the general bounded fragment of first-order logic.
- The semantic characterisation theorem makes essential use of bidirectional modal logic.
- The bounded $q2^q$ in the semantic characterisation theorem is not known to be sharp.