Team Semantics and Independence Notions in Quantum Physics

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Outline

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Team Semantics

Given a structure $\mathfrak{A}$ and a finite set V of variables, a team of $\mathfrak{A}$ is a set X of assignments $s\colon V \to A$.

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	x	y	z
s_0	0	0	0
s_1	0	0	1
s_2	0	1	0

Team Semantics (cont.)

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- $\mathfrak{A} \models_X =(\vec{x}, \vec{y})$ if

$$\forall s, s' \in X(s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})).$$

Team Semantics (cont.)

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$$\forall s, s' \in X(s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})).$$

- $\mathfrak{A} \models_X \vec{x} \perp_{\vec{y}} \vec{z}$ if

$$\forall s, s' \in X\big(s(\vec{y}) = s'(\vec{y}) \implies \\ \exists s'' \in X\big(s''(\vec{x}\vec{y}) = s(\vec{x}\vec{y}) \wedge s''(\vec{z}) = s'(\vec{z})\big)\big).$$

Team Semantics (cont.)

- $\mathfrak{A} \models_X =\!(\vec{x}, \vec{y})$ if

$$\forall s, s' \in X(s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})).$$

- $\mathfrak{A} \models_X \vec{x} \perp_{\vec{y}} \vec{z}$ if

$$\forall s, s' \in X\big(s(\vec{y}) = s'(\vec{y}) \implies$$
$$\exists s'' \in X\big(s''(\vec{x}\vec{y}) = s(\vec{x}\vec{y}) \wedge s''(\vec{z}) = s'(\vec{z})\big)\big).$$

- $=\!(\vec{x}, \vec{y}) \equiv \vec{y} \perp_{\vec{x}} \vec{y}$.

Team Semantics (cont.)

- $\mathfrak{A} \models_X \varphi$ if $\mathfrak{A} \models_s \varphi$ for all $s \in X$, whenever φ is a first-order atomic or negated atomic formula.
- $\mathfrak{A} \models_X \varphi \wedge \psi$ if $\mathfrak{A} \models_X \varphi$ and $\mathfrak{A} \models_X \psi$.
- $\mathfrak{A} \models_X \varphi \vee \psi$ if $\mathfrak{A} \models_Y \varphi$ and $\mathfrak{A} \models_Z \psi$ for some $X, Y \subseteq X$ such that $X = Y \cup Z$.
- $\mathfrak{A} \models_X \exists x\varphi$ if $\mathfrak{A} \models_{X[F/x]} \varphi$ for some function $F\colon X \to \mathcal{P}(A) \setminus \{\emptyset\}$, where $X[F/x] = \{s(a/x) \mid s \in X, a \in F(s)\}$.
- $\mathfrak{A} \models_X \forall x\varphi$ if $\mathfrak{A} \models_{X[A/x]} \varphi$, where $X[A/x] = \{s(a/x) \mid s \in X, a \in A\}$.

Probabilistic Team Semantics [Durand et al. 2018]

Given a finite structure $\mathfrak{A}$ and a set V of variables, a probabilistic team of $\mathfrak{A}$ is a probability distribution $\mathbb{X}\colon A^V \to [0,1]$. V is called the variable domain of $\mathbb{X}$ and A the value domain of $\mathbb{X}$.

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	x	y	z	probability
s_0	0	0	0	0.2
s_1	0	0	1	0.3
s_2	0	1	0	0.5
s_3	0	1	1	0

	x	y	z	probability
s_4	1	0	0	0
s_5	1	0	1	0
s_6	1	1	0	0
s_7	1	1	1	0

Probabilistic Team Semantics (cont.)

- $\mathfrak{A} \models_{\mathbb{X}} \vec{x} \perp\!\!\!\perp_{\vec{y}} \vec{z}$ if for all $\vec{a} \in A^{|\vec{x}|}$, $\vec{b} \in A^{|\vec{y}|}$ and $\vec{c} \in A^{|\vec{z}|}$,

$$\left|\mathbb{X}_{\vec{x}\vec{y}=\vec{a}\vec{b}}\right| \cdot \left|\mathbb{X}_{\vec{y}\vec{z}=\vec{b}\vec{c}}\right| = \left|\mathbb{X}_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{c}}\right| \cdot \left|\mathbb{X}_{\vec{y}=\vec{b}}\right|,$$

where $\left|\mathbb{X}_{\vec{v}=\vec{d}}\right| = \sum_{\substack{s \in \operatorname{supp} \mathbb{X} \\ s(\vec{v})=\vec{d}}} \mathbb{X}(s)$.

Probabilistic Team Semantics (cont.)

- $\mathfrak{A} \models_X \vec{x} \perp\!\!\!\perp_{\vec{y}} \vec{z}$ if for all $\vec{a} \in A^{|\vec{x}|}$, $\vec{b} \in A^{|\vec{y}|}$ and $\vec{c} \in A^{|\vec{z}|}$,

$$\left|\mathbb{X}_{\vec{x}\vec{y}=\vec{a}\vec{b}}\right| \cdot \left|\mathbb{X}_{\vec{y}\vec{z}=\vec{b}\vec{c}}\right| = \left|\mathbb{X}_{\vec{x}\vec{y}\vec{z}=\vec{a}\vec{b}\vec{c}}\right| \cdot \left|\mathbb{X}_{\vec{y}=\vec{b}}\right|,$$

where $\left|\mathbb{X}_{\vec{v}=\vec{d}}\right| = \sum_{\substack{s \in \operatorname{supp} \mathbb{X} \\ s(\vec{v})=\vec{d}}} \mathbb{X}(s)$.

- $\mathfrak{A} \models_{\mathbb{X}} =\!(\vec{x}, \vec{y})$ if $\mathfrak{A} \models_{\mathbb{X}} \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{y}$.

Probabilistic Team Semantics (cont.)

For probabilistic teams $\mathbb{X}$ and $\mathbb{Y}$ and $r \in [0, 1]$, define

$$(\mathbb{X} \sqcup_r \mathbb{Y})(s) := r\mathbb{X}(s) + (1 - r)\mathbb{Y}(s).$$

Probabilistic Team Semantics (cont.)

For probabilistic teams $\mathbb{X}$ and $\mathbb{Y}$ and $r \in [0,1]$, define

$$(\mathbb{X} \sqcup_r \mathbb{Y})(s) := r\mathbb{X}(s) + (1-r)\mathbb{Y}(s).$$

Given a function F from the set X to the set of all probability distributions on A, define

$$\mathbb{X}[F/v](s(a/v)) := \sum_{\substack{t \in \operatorname{supp} \mathbb{X} \\ t(a/v) = s(a/v)}} \mathbb{X}(t)F(t)(a).$$

Probabilistic Team Semantics (cont.)

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$$(\mathbb{X} \sqcup_r \mathbb{Y})(s) := r\mathbb{X}(s) + (1-r)\mathbb{Y}(s).$$

Given a function F from the set X to the set of all probability distributions on A, define

$$\mathbb{X}[F/v](s(a/v)) := \sum_{\substack{t \in \operatorname{supp} \mathbb{X} \\ t(a/v) = s(a/v)}} \mathbb{X}(t)F(t)(a).$$

$\mathbb{X}[A/v] := \mathbb{X}[F/v]$ for $F : s \mapsto 1/|A|$.

Probabilistic Team Semantics (cont.)

- $\mathfrak{A} \models_{\mathbb{X}} \alpha$ for a first-order atomic or negated atomic formula α if $\mathfrak{A} \models_X \alpha$, where X is the possibilistic collapse of $\mathbb{X}$.
- $\mathfrak{A} \models_{\mathbb{X}} \varphi \wedge \psi$ if $\mathfrak{A} \models_{\mathbb{X}} \varphi$ and $\mathfrak{A} \models_{\mathbb{X}} \psi$.
- $\mathfrak{A} \models_{\mathbb{X}} \varphi \vee \psi$ if $\mathfrak{A} \models_{\mathbb{Y}} \varphi$ and $\mathfrak{A} \models_{\mathbb{Z}} \psi$ for some probabilistic teams $\mathbb{Y}$ and $\mathbb{Z}$, and $r \in [0,1]$ such that $\mathbb{X} = \mathbb{Y} \sqcup_r \mathbb{Z}$.
- $\mathfrak{A} \models_{\mathbb{X}} \exists v \varphi$ if $\mathfrak{A} \models_{\mathbb{X}[F/v]} \varphi$ for some function $F\colon \operatorname{supp} \mathbb{X} \to \{p \in [0,1]^{A_{\mathfrak{s}(v)}} \mid p \text{ is a distribution}\}$.
- $\mathfrak{A} \models_{\mathbb{X}} \forall v \varphi$ if $\mathfrak{A} \models_{\mathbb{X}[A]} \varphi$.

Hidden-Variable Models of Quantum Mechanics

- Could the non-deterministic nature of quantum mechanics be explained by including "hidden" variables in the models?
- Brandenburger & Yanofsky: a purely probabilistic framework
- Abramsky: a relational (possibilistic) framework

Empirical & Hidden-Variable Teams

We consider variables of three sorts:

- $V_\mathrm{m} = \{x_0, \ldots, x_{n-1}\}$ (“measurement variables”),
- $V_\mathrm{o} = \{y_0, \ldots, y_{n-1}\}$ (“outcome variables”), and
- $V_\mathrm{h} = \{z_0, \ldots, z_{l-1}\}$ (“hidden variables”).

X is an *empirical team* if $\mathrm{dom}(X) = V_\mathrm{m} \cup V_\mathrm{o}$.

X is a *hidden-variable team* if $\mathrm{dom}(X) = V_\mathrm{m} \cup V_\mathrm{o} \cup V_\mathrm{h}$.

Empirical & Hidden-Variable Teams (cont.)

x_0	y_0	$\dots$	x_{n-1}	y_{n-1}	z_0	$\dots$	z_{l-1}
a_0^0	b_0^0	$\dots$	a_{n-1}^0	b_{n-1}^0	γ_0^0	$\dots$	γ_{l-1}^0
a_0^2	b_0^1	$\dots$	a_{n-1}^2	b_{n-1}^1	γ_0^1	$\dots$	γ_{l-1}^2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_0^{m-1}	b_0^{m-1}	$\dots$	a_{n-1}^{m-1}	b_{n-1}^{m-1}	γ_0^{m-1}	$\dots$	γ_{l-1}^{m-1}

Properties of Empirical Teams

Weak Determinism: "the outcomes of the measurements are completely determined"

$$=(\vec{x}, \vec{y})$$

No-Signalling: "the choice of measurement by one party cannot be signalled to the other parties".

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i} y_i$$

Properties of Hidden-Variable Teams

Strong Determinism: "the outcome of each individual measurement is completely determined by that measurement (and the hidden variable) alone"

$$\bigwedge_{i<n} {=}(x_i\vec{z}, y_i)$$

z-Independence: "the value of the hidden variable is independent of the choice of measurements"

$$\vec{z} \perp \vec{x}$$

Parameter Independence: a hidden-variable version of no-signalling

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i$$

Relationships between the Properties

Strong determinism implies parameter independence

$$=\!(x_i\vec{z}, y_i) \vdash \{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i$$

Empirical vs. Hidden-Variable Teams

An empirical team supports no-signalling iff it can be realized by a hidden-variable team supporting z-independence and parameter independence.

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In other words, the following formulas are equivalent.

1. $\bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i} y_i$,
2. $\tilde{\exists} z_0 \exists z_1 \ldots \exists z_{l-1} \left(\vec{z} \perp \vec{x} \wedge \bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i\right)$.

Empirical vs. Hidden-Variable Teams

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$\mathfrak{A} \models_X \tilde{\exists} x \varphi \iff \mathfrak{B} \models_X \exists x \varphi$ for some expansion $\mathfrak{B}$ of $\mathfrak{A}$ by the sort of x. [Väänänen 2014]

Probabilistic Empirical & Hidden-Variable Teams

We say that a probabilistic team $\mathbb{X}$ is empirical if the variable domain of $\mathbb{X}$ is $V_{\mathsf{m}} \cup V_{\mathsf{o}}$. We say that $\mathbb{X}$ is a probabilistic hidden-variable team if the variable domain is $V_{\mathsf{m}} \cup V_{\mathsf{o}} \cup V_{\mathsf{h}}$.

Properties of Probabilistic Teams

Parameter independence for ordinary teams:

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i$$

Parameter independence for probabilistic teams:

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp\!\!\!\perp_{x_i \vec{z}} y_i$$

The team $X = \{s \in A^V \mid \mathbb{X}(s) > 0\}$ is called the *possibilistic collapse* of $\mathbb{X}$.

Proposition

1. *For any* $\varphi \in \mathrm{FO}(\perp\!\!\!\perp)$, *if* $\mathbb{X} \models \varphi$, *then* $X \models \varphi$.
2. *For any* $\varphi \in \mathrm{FO}(=\!(\cdot))$, *if* $X \models \varphi$, *then* $\mathbb{X} \models \varphi$.

Axioms of Independence Atom

The axioms of independence atom [Grädel & Väänänen 2013] are the following.

1. $\vec{y} \perp_{\vec{x}} \vec{y}$ entails $\vec{y} \perp_{\vec{x}} \vec{z}$. (Constancy Rule)
2. $\vec{x} \perp_{\vec{x}} \vec{y}$. (Reflexivity Rule)
3. $\vec{z} \perp_{\vec{x}} \vec{y}$ entails $\vec{y} \perp_{\vec{x}} \vec{z}$. (Symmetry Rule)
4. $\vec{y}y' \perp_{\vec{x}} \vec{z}z'$ entails $\vec{y} \perp_{\vec{x}} \vec{z}$. (Weakening Rule)
5. If $\vec{z'}$, $\vec{x'}$ and $\vec{y'}$ are permutations of $\vec{z}$, $\vec{x}$ and $\vec{y}$ respectively, then $\vec{y} \perp_{\vec{x}} \vec{z}$ entails $\vec{y'} \perp_{\vec{x'}} \vec{z'}$. (Permutation Rule)
6. $\vec{z} \perp_{\vec{x}} \vec{y}$ entails $\vec{y}\vec{x} \perp_{\vec{x}} \vec{z}\vec{x}$. (Fixed Parameter Rule)
7. $\vec{x} \perp_{\vec{z}} \vec{y} \wedge \vec{u} \perp_{\vec{z}\vec{x}} \vec{y}$ entails $\vec{u} \perp_{\vec{z}} \vec{y}$. (First Transitivity Rule)
8. $\vec{y} \perp_{\vec{z}} \vec{y} \wedge \vec{z}\vec{x} \perp_{\vec{y}} \vec{u}$ entails $\vec{x} \perp_{\vec{z}} \vec{u}$. (Second Transitivity Rule)

Axioms Are Valid in Probabilistic Team Semantics

Everything that is provable from the axioms is true also for probabilistic teams:

Proposition

The probabilistic independence atom satisfies the axioms of the independence atom.

Proof idea: The axioms of the independence atom follow from the so called *separoid axioms* [Dawid 2001], and the probabilistic independence atom can be realized as a separoid.

Separoids

Definition

Let A be a set, $\leq$ a binary relation on A and $\perp\!\!\!\perp$ a ternary relation on A. The structure $(A, \leq, \perp\!\!\!\perp)$ is a *separoid* if

1. $(A, \leq)$ is a quasiorder such that for any $a, b \in A$, the set $\{a, b\}$ has a least upper bound $a \vee b \in A$, and
2. the following axioms hold for all $a, b, c, d \in A$:
   - (P1) $a \perp\!\!\!\perp_c b$ implies $b \perp\!\!\!\perp_c a$.
   - (P2) $a \perp\!\!\!\perp_a b$.
   - (P3) $a \perp\!\!\!\perp_c b$ and $d \leq b$ implies $a \perp\!\!\!\perp_c d$.
   - (P4) $a \perp\!\!\!\perp_c b$ and $d \leq b$ implies $a \perp\!\!\!\perp_{c \vee d} b$.
   - (P5) $a \perp\!\!\!\perp_c b$ and $a \perp\!\!\!\perp_{b \vee c} d$ implies $a \perp\!\!\!\perp_c (b \vee d)$.

Proposition

Let V be a finite set of variables and $\mathbb{X}$ a probabilistic team with variable domain V. Then the structure $(V^{<\omega}, \preceq, \perp\!\!\!\perp)$ is a separoid, where

$$\vec{x} \preceq \vec{y} \iff \mathbb{X} \models =(\vec{y}, \vec{x})$$

and

$$\vec{x} \perp\!\!\!\perp_{\vec{z}} \vec{y} \iff \mathbb{X} \models \vec{x} \perp\!\!\!\perp_{\vec{z}} \vec{y}.$$

Building Probabilistic Teams

When does a formula φ of (ordinary) independence logic have the property

$$X \models \varphi \implies \exists \mathbb{X}(\mathbb{X} \models \varphi \text{ and } \mathbb{X} \text{ collapses to } X)?$$

Building Probabilistic Teams

When does a formula φ of (ordinary) independence logic have the property

$$X \models \varphi \implies \exists \mathbb{X}(\mathbb{X} \models \varphi \text{ and } \mathbb{X} \text{ collapses to } X)?$$

Define a new operation PR on formulas, such that an ordinary team X satisfies $\mathrm{PR}\,\varphi$ if it is the possibilistic collapse of some probabilistic team $\mathbb{X}$ such that $\mathbb{X} \models \varphi$.

Building Probabilistic Teams (cont.)

Proposition

There is an empirical team X supporting no-signalling such that no probabilistic team $\mathbb{X}$ that supports probabilistic no-signalling and collapses to X, i.e.

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i} y_i \quad \not\models \quad \mathrm{PR} \bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i} y_i .$$

Proof sketch.

The following team cannot be assigned probabilities so that probabilistic no-signalling is satisfied.

	x_0	x_1	y_0	y_1
s_0	0	0	0	0
s_1	0	0	0	1
s_2	0	0	1	1
s_3	0	1	0	0
s_4	0	1	1	0
s_5	0	1	1	1

	x_0	x_1	y_0	y_1
s_6	1	0	0	0
s_7	1	0	1	0
s_8	1	0	1	1
s_9	1	1	0	0
s_{10}	1	1	0	1
s_{11}	1	1	1	1

□

Building Probabilistic Teams (cont.)

Proposition

Let X be a hidden-variable team supporting measurement locality, z-independence and locality. Then there is a probabilistic hidden-variable team $\mathbb{X}$ supporting probabilistic measurement locality, probabilistic z-independence and probabilistic locality whose possibilistic collapse is X. In other words, the formula

$$\varphi := \vec{z} \perp \vec{x} \wedge \bigwedge_{i<n} x_i y_i \perp_{\vec{z}} \{x_j y_j \mid j \neq i\}$$

is such that $\varphi \models \mathrm{PR}\,\varphi$.

Proof sketch.

Define

$$\mathrm{Prob}\,(X)\,(s) = \begin{cases} 1/(m_{\mathrm{h}} m_{\mathrm{m}} m_{\mathrm{o}}(s(\vec{x}), s(\vec{z}))) & \text{if } s(\vec{x}) \in M,\ s(\vec{z}) \in \Gamma, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Gamma = \{s(\vec{z}) \mid s \in X\},$$
$$M = \{s(\vec{x}) \mid s \in X\}, \text{ and}$$
$$O_{\vec{a},\vec{\gamma}} = \{s(\vec{y}) \mid s \in X, s(\vec{x}\vec{z}) = \vec{a}\vec{\gamma}\},$$

and $m_{\mathrm{h}} = |\Gamma|$, $m_{\mathrm{m}} = |M|$ and $m_{\mathrm{o}}(\vec{a}, \vec{\gamma}) = \left|O_{\vec{a},\vec{\gamma}}\right|$. Then $\mathrm{Prob}\,(X)$ supports the given properties and collapses to X. □

No-Go Theorems

There is an empirical team that cannot be realized by any hidden-variable team supporting single-valuedness and outcome independence.

No-Go Theorems

There is an empirical team that cannot be realized by any hidden-variable team supporting single-valuedness and outcome independence.

Theorem

$\tilde{\exists} z_0 \exists z_1 \ldots \exists z_{l-1} \left(=\!(\vec{z}) \wedge \bigwedge_{i<n} y_i \perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right)$ *is not valid.*

Proof.

As demonstrated, for instance, by the following team.

	x_0	x_1	y_0	y_1
s	0	1	0	1
s'	0	1	1	0

We call the above the *EPR team*. □

No-Go Theorems (cont.)

There is an empirical team that cannot be realized by a hidden-variable team supporting z-independence and locality.

Theorem

$$\tilde{\exists} z_0 \exists z_1 \ldots \exists z_{l-1} \left(\vec{z} \perp \vec{x} \wedge \bigwedge_{i<n} \left(\left(\{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i \right) \wedge \left(y_i \perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right) \right) \right)$$

is not valid.

Proof.

As is demonstrated, for instance, by the following team.

	x_0	x_1	x_2	y_0	y_1	y_2
s_0	0	0	0	0	0	1
s_1	0	0	0	0	1	0
s_2	0	0	0	1	0	0
s_3	0	0	0	1	1	1
s_4	0	1	1	0	0	0
s_5	0	1	1	0	1	1
s_6	1	0	1	1	0	1
s_7	1	1	0	1	1	0

This is an example of a *GHZ team*. □

Quantum-Mechanical Teams

Definition

A probabilistic empirical team $\mathbb{X}$ is *quantum-mechanical* if it represents the probability distribution of measurement outcomes in a finite-dimensional quantum system.

Quantum-Mechanical Teams

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A probabilistic empirical team $\mathbb{X}$ is *quantum-mechanical* if it represents the probability distribution of measurement outcomes in a finite-dimensional quantum system.

Define a new atomic formula QR such that an ordinary team X satisfies QR if X is the possibilistic collapse of a quantum-mechanical team.

Definition

Let M and O be sets of n-tuples, and denote $M_i = \{a_i \mid \vec{a} \in M\}$ and $O_i = \{b_i \mid \vec{b} \in O\}$. A quantum system of type (M, O) is a tuple $(\mathcal{H}, (A_i^{a,b})_{a \in M_i, b \in O_i, i<n}, \rho)$, where

- $\mathcal{H}$ is the tensor product $\bigotimes_{i<n} \mathcal{H}_i$ of finite-dimensional Hilbert spaces $\mathcal{H}_i$, $i < n$,
- for all $i < n$ and $a \in M_i$, $\{A_i^{a,b} \mid b \in O_i\}$ is a positive operator-valued measure on $\mathcal{H}_i$, and
- ρ is a density operator on $\mathcal{H}$, i.e. $\rho = \sum_{j<k} p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|$, where $\left|\psi_j\right\rangle$ is a unit vector of $\mathcal{H}$ and $p_j \in [0, 1]$ for all $j < k$ and $\sum_{j<k} p_j = 1$.

For each measurement $\vec{a} \in M$, we define the probability distribution $p_{\vec{a}}$ of outcomes by setting $p_{\vec{a}}(\vec{b}) := \mathrm{Tr}(A^{\vec{a},\vec{b}}\rho)$, where $A^{\vec{a},\vec{b}}$ denotes the operator $\bigotimes_{i<n} A_i^{a_i,b_i}$.

Definition

Let $\mathbb{X}$ be a probabilistic team with variable domain $V_{\mathrm{m}} \cup V_{\mathrm{o}}$ and denote $M = \{s(\vec{x}) \mid s \in \operatorname{supp} \mathbb{X}\}$ and $O = \{s(\vec{y}) \mid s \in \operatorname{supp} \mathbb{X}\}$. We say that $\mathbb{X}$ is *quantum-mechanical* if there exists a quantum system

$$(\mathcal{H}, (A_i^{a,b})_{a \in M_i, b \in O_i, i<n}, \rho)$$

of type (M, O) such that for all assignments s, we have $\mathbb{X}(s) = p_{s(\vec{x})}(s(\vec{y}))/\left|M\right|$. We call a quantum-mechanical team $\mathbb{X}$ a *quantum realization* of an empirical team X if X is the possibilistic collapse of $\mathbb{X}$.

Quantum-Mechanical Teams (cont.)

Proposition

1. *The EPR team is a collapse of a quantum-mechanical team, hence*

$$\mathrm{QR} \not\models \exists \vec{z} \left(=\!(\vec{z}) \wedge \bigwedge_{i<n} y_i \perp\!\!\!\perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right).$$

2. *A GHZ team is a collapse of a quantum-mechanical team, hence*

$$\mathrm{QR} \not\models \exists \vec{z} \left(\vec{z} \perp\!\!\!\perp \vec{x} \wedge \bigwedge_{i<n} \{x_j \mid j \neq i\} \perp\!\!\!\perp_{x_i \vec{z}} y_i \wedge \bigwedge_{i<n} y_i \perp\!\!\!\perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right).$$

Proposition

The set $\{X \mid X \models \mathrm{QR}\}$ *is undecidable but recursively enumerable.*

Proof idea: There is a many-one reduction from two-player one-round non-local games that have a perfect quantum strategy to teams that have a quantum realization.

Determining whether a non-local game has a perfect quantum strategy is undecidable. [Slofstra 2019]

Open Questions

- Properties of PR?
- Properties of QR?
- (Un)decidability results for the above?
- Definability in some logic?

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