Arboreal categories

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Recap on Game Comonads

- Model comparison games (Ehrenfeucht-Fraïssé, pebble, bisimulation games, etc.) yield semantic characterisations of logical equivalences:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} \iff \forall \varphi \in \mathcal{L}. \ (\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi)$$

- If $\mathcal{L} = \mathrm{FO}$, then $\equiv^{\mathcal{L}}$ coincides with elementary equivalence (cf. Keisler-Shelah theorem). In general, we are interested in resource-bounded fragments of FO.

- Game comonads arise from the idea that model comparison games should be seen as semantic constructions in their own right.

- Category theoretic view on resource-sensitive Model Theory. (Or *Model Theory without Compactness*.)

The Ehrenfeucht-Fraïssé comonad $\mathbb{E}_k$ (see board)

The right adjoint is uniquely determined by the forgetful functor, and the comonad by the adjunction. Moreover, the adjunction is comonadic, hence the category of coalgebras for $\mathbb{E}_k$ is exactly $\mathcal{R}_k^E(\sigma)$.

This gives us:

- The Ehrenfeucht-Fraïssé game
- Equivalences of structures induced by:
  - The fragment FO_k of FO consisting of sentences with quantifier rank $\leq k$
  - The existential positive fragment $\exists^+\mathrm{FO}_k$ of FO_k
  - The extension $\mathrm{FO}_k(\#)$ of FO_k with counting quantifiers
- We also recover the combinatorial parameter of tree-depth from the coalgebras of the comonad

This template can be used to give similar analyses of a wealth of other logical and combinatorial notions (e.g. pebble and bisimulation games).

Arboreal categories

Abramsky and Reggio. *Arboreal categories and resources*. ICALP'21.
Abramsky and Reggio. *Arboreal categories: An axiomatic theory of resources*. (`arXiv:2102.08109`)

Paths

Let $\mathcal{C}$ be a category equipped with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$. Arrows in $\mathcal{Q}$ are called quotients and arrows in $\mathcal{M}$ embeddings.

Definition

An object X of $\mathcal{C}$ is called a path provided the poset $\mathbb{S}\,X$ of $\mathcal{M}$-subobjects of X is a finite chain. Paths will be denoted by $P, Q, \ldots$

Example

Paths in $\mathcal{R}_k^E(\sigma)$ are the forest-ordered structures $(A, \leq)$ such that $\leq$ is a linear order of cardinality at most k.
($\mathcal{Q}$ = surjective morphisms, $\mathcal{M}$ = morphisms that are σ-embeddings)

Arboreal categories defined

Definition

An arboreal category is a category $\mathcal{C}$, equipped with a stable proper factorisation system, that satisfies the following conditions:

1. $\mathcal{C}$ has all coproducts of small families of paths.
2. For any paths P, Q, Q' in $\mathcal{C}$, if a composite $P \to Q \to Q'$ is a quotient, then so is $P \to Q$.
3. Every object of $\mathcal{C}$ is the colimit of its path embeddings.
4. Every path in $\mathcal{C}$ is connected.

Example

$\mathcal{R}_k^E(\sigma)$ is arboreal. Similarly, the categories of coalgebras for the game comonads $\mathbb{P}_k$ and $\mathbb{M}_k$ are arboreal.

The functor of paths

Given any object X of $\mathcal{C}$, we let $\mathbb{P}\,X$ denote the sub-poset of $\mathbb{S}\,X$ consisting of the path embeddings.

Theorem

*Let $\mathcal{C}$ be an arboreal category. Then the assignment $X \mapsto \mathbb{P}\,X$ induces a functor $\mathbb{P}\colon \mathcal{C} \to$ **Trees** into the category of trees.*

Some useful properties of paths:

Proposition

The following statements hold in any arboreal category $\mathcal{C}$:

1. *Paths are closed under quotients and embeddings.*
2. *Between any two paths there is at most one embedding.*
3. $\forall X \in \mathcal{C}$, $\mathbb{S}\,X$ *is isomorphic to the lattice of downsets of* $\mathbb{P}\,X \setminus \{\bot\}$.

Pathwise embeddings and open morphisms

Definition

Let $f\colon X \to Y$ be a morphism in an arboreal category $\mathcal{C}$.

1. f is a pathwise embedding if, for all path embeddings $m\colon P \rightarrowtail X$, the composite $f \circ m\colon P \to Y$ is a path embedding.
2. f is open if it satisfies the following path-lifting property: Given any commutative square

$$\begin{array}{ccc} P & \rightarrowtail & Q \\ \downarrow & \swarrow & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

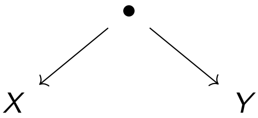
   with P, Q paths, there is $Q \to X$ making the two triangles commute.

Joyal, Nielson, and Winskel. *Bisimulation and open maps*. LiCS'93.

Bisimilarity

Definition

Objects X, Y of an arboreal category $\mathcal{C}$ are bisimilar is there exists a span of open pathwise embeddings connecting them:



Using the functor $\mathbb{P}\colon \mathcal{C} \to \mathbf{Trees}$ we can define a back-and-forth game $\mathcal{G}(X, Y)$ that captures bisimilarity:

Theorem

Suppose that X and Y admit a product in $\mathcal{C}$. Then they are bisimilar if, and only if, Duplicator has a winning strategy in the game $\mathcal{G}(X, Y)$.

Games in arboreal categories

The back-and-forth game $\mathcal{G}(X, Y)$ is played by Spoiler and Duplicator on the trees $\mathbb{P}X$ and $\mathbb{P}Y$ as follows:

- Positions in the game are pairs $(m, n) \in \mathbb{P}X \times \mathbb{P}Y$.
- The winning relation $\mathcal{W}(X, Y) \subseteq \mathbb{P}X \times \mathbb{P}Y$ consists of the pairs (m, n) such that $\text{dom}(m) \cong \text{dom}(n)$.
- Let $\bot_X : P \rightarrowtail X$ and $\bot_Y : Q \rightarrowtail Y$ be the roots of $\mathbb{P}X$ and $\mathbb{P}Y$, respectively. If $P \ncong Q$, then Duplicator loses the game. Otherwise, the initial position is $(\bot_X, \bot_Y)$.
- At the start of each round, the position is specified by a pair $(m, n) \in \mathbb{P}X \times \mathbb{P}Y$, and the round proceeds as follows: Either Spoiler chooses some $m' \succ m$ and Duplicator must respond with some $n' \succ n$, or Spoiler chooses some $n'' \succ n$ and Duplicator must respond with $m'' \succ m$.
- Duplicator wins the round if they are able to respond and the new position is in $\mathcal{W}(X, Y)$. Duplicator wins the game if they win the k-round game for every $k \geq 0$.

Resource indexing

Let $\mathcal{C}$ be an arboreal category, with full subcategory of paths $\mathcal{C}_p$.
$\mathcal{C}$ is resource-indexed by a resource parameter k if for all $k > 0$, there is a full subcategory $\mathcal{C}_p^k$ of $\mathcal{C}_p$ closed under embeddings with

$$\mathcal{C}_p^1 \hookrightarrow \mathcal{C}_p^2 \hookrightarrow \mathcal{C}_p^3 \hookrightarrow \cdots$$

This induces a corresponding tower of full subcategories $\mathcal{C}_k$ of $\mathcal{C}$, with the objects of $\mathcal{C}_k$ those generated by the paths in $\mathcal{C}_p^k$.

Definition

Let $\{\mathcal{C}_k\}$ be a resource-indexed arboreal category. A resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$ is a family of adjunctions

$$\mathcal{C}_k \underset{R_k}{\overset{L_k}{\rightleftarrows}} \mathcal{E}\,. \quad (L_k \dashv R_k)$$

Resource-indexed relations

Every resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$ induces resource-indexed relations $\to_k^{\mathcal{C}}$, $\leftrightarrow_k^{\mathcal{C}}$ and $\cong_k^{\mathcal{C}}$ on $\mathcal{E}$.

- $a \to_k^{\mathcal{C}} b$ if there exists a morphism $R_k a \to R_k b$ in $\mathcal{C}_k$.
- $a \leftrightarrow_k^{\mathcal{C}} b$ if $R_k a$ and $R_k b$ are bisimilar in $\mathcal{C}_k$.
- $a \cong_k^{\mathcal{C}} b$ if $R_k a$ and $R_k b$ are isomorphic in $\mathcal{C}_k$.

Example

For the Ehrenfeucht-Fraïssé resource-indexed arboreal adjunction,

- $\to_k^{\mathcal{C}}$ coincides with $\Rrightarrow^{\exists^+\mathrm{FO}_k}$.
- $\leftrightarrow_k^{\mathcal{C}}$ coincides with $\equiv^{\mathrm{FO}_k}$.
- $\cong_k^{\mathcal{C}}$ coincides with $\equiv^{\mathrm{FO}_k(\#)}$.

(FO_k is first-order logic with quantifier rank at most k.)

And similarly for pebble, bisimulation games, etc.

Homomorphism Preservation Theorems

Abramsky and Reggio. *Arboreal categories and homomorphism preservation theorems*. In preparation.

Łoś, Lyndon and Tarski

- Homomorphism preservation theorems relate the syntactic shape of a sentence with the semantic property of being preserved under (various classes of) homomorphisms between structures.
- A first-order sentence φ in a (relational) vocabulary σ is said to be preserved under homomorphisms if, whenever there is a homomorphism of σ-structures $\mathcal{A} \to \mathcal{B}$, $\mathcal{A} \vDash \varphi$ entails $\mathcal{B} \vDash \varphi$.

Theorem (Łoś, Lyndon and Tarski, 1950s)

A first-order sentence φ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence ψ.

The Equirank HPT

- The HPT is a fairly straightforward consequence of the compactness theorem for first-order logic.
- Ineffective approach if we want to determine to which extent the passage from φ to ψ increases the "complexity" of the former.
- One way to measure the complexity of a formula is in terms of its quantifier rank, *i.e.* the maximum number of nested quantifiers appearing in the formula.

Theorem (Rossman, 2007)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

The Finite HPT

The Equirank HPT is the "first step" in the proof of Rossman's celebrated Finite HPT:

Theorem (Rossman, 2007)

A first-order sentence is preserved under homomorphisms between finite structures if, and only if, it is equivalent over finite structures to an existential positive sentence.

Model classes

Fix a resource-indexed arboreal adjunction between $\mathcal{R}(\sigma)$ and $\mathcal{C}$.

Lemma

Let $\mathcal{L}_k$ be a finite Boolean subalgebra of FO such that $\leftrightarrow_k^{\mathcal{C}} = \equiv^{\mathcal{L}_k}$. The following are equivalent for any full subcategory $\mathcal{D}$ of $\mathcal{R}(\sigma)$:

1. $\mathcal{D} = \mathbf{Mod}(\varphi)$ *for some* $\varphi \in \mathcal{L}_k$.
2. $\mathcal{D}$ *is saturated under* $\leftrightarrow_k^{\mathcal{C}}$*, i.e. for all* σ*-structures* $\mathcal{A}, \mathcal{B}$*, if* $\mathcal{A} \in \mathcal{D}$ *and* $\mathcal{A} \leftrightarrow_k^{\mathcal{C}} \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{D}$.

Lemma

Let $\mathcal{L}_k$ be a finite sublattice of FO such that $\rightarrow_k^{\mathcal{C}} = \Rrightarrow^{\mathcal{L}_k}$. The following are equivalent for any full subcategory $\mathcal{D}$ of $\mathcal{R}(\sigma)$:

1. $\mathcal{D} = \mathbf{Mod}(\psi)$ *for some* $\psi \in \mathcal{L}_k$.
2. $\mathcal{D}$ *is upwards closed with respect to* $\rightarrow_k^{\mathcal{C}}$*, i.e. for all* σ*-structures* $\mathcal{A}, \mathcal{B}$*, if* $\mathcal{A} \in \mathcal{D}$ *and* $\mathcal{A} \rightarrow_k^{\mathcal{C}} \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{D}$.

(HP) and (HP$^\#$)

Fix an arbitrary resource-indexed arboreal adjunction between an extensional category $\mathcal{E}$ and a resource-indexed arboreal category $\mathcal{C}$. For all $k > 0$, consider the following statement:

(HP) For any full subcategory $\mathcal{D}$ of $\mathcal{E}$ saturated under $\leftrightarrow_k^{\mathcal{C}}$, $\mathcal{D}$ is closed under morphisms iff it is upwards closed with respect to $\rightarrow_k^{\mathcal{C}}$.

Example

For the Ehrenfeucht-Fraïssé resource-indexed arboreal adjunction, (HP) is precisely Rossman's equirank HPT.

Replacing $\leftrightarrow_k^{\mathcal{C}}$ with $\cong_k^{\mathcal{C}}$, we obtain a strengthening of (HP):

(HP$^\#$) For any full subcategory $\mathcal{D}$ of $\mathcal{E}$ saturated under $\cong_k^{\mathcal{C}}$, $\mathcal{D}$ is closed under morphisms iff it is upwards closed with respect to $\rightarrow_k^{\mathcal{C}}$.

Note: the "if" parts of (HP) and (HP$^\#$) always hold.

The bisimilar companion property

Definition

A resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$, with induced comonads G_k on $\mathcal{E}$, has the bisimilar companion property if $a \leftrightarrow_k^{\mathcal{C}} G_k a$ for all $a \in \mathcal{E}$ and all $k > 0$.

Proposition

Consider any resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$ with the bisimilar companion property. Then (HP) *holds.*

Proof.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{E}$ saturated under $\leftrightarrow_k^{\mathcal{C}}$ and closed under morphisms. We must prove that, if $a \to_k^{\mathcal{C}} b$ and $a \in \mathcal{D}$, then also $b \in \mathcal{D}$. See whiteboard. □

Example

The guarded comonads $\mathbb{G}_k$, introduced in

- S. Abramsky and D. Marsden, Comonadic semantics for guarded fragments, LiCS 2021

have the bisimilar companion property. So, we get an "equi-resource" homomorphism preservation theorem for guarded logics.

Next, we look at a strengthening of the bisimilar companion property.

Idempotency

Definition

A resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$ is idempotent if so are the induced comonads G_k, *i.e.* $\delta_a \colon G_k a \to G_k G_k a$ is an isomorphism for all $a \in \mathcal{E}$ and all $k > 0$.

Proposition

Consider any idempotent resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$. Then ($\mathrm{HP}^{\#}$) *holds.*

Proof.

If G_k is idempotent then, for all $a \in \mathcal{E}$, we have $a \cong_k^{\mathcal{C}} G_k a$. The statement follows by reasoning as in the previous proposition. □

Example

The modal comonads $\mathbb{M}_k$ on pointed Kripke structures are idempotent. Thus, we obtain the following "equidepth" homomorphism preservation theorem for (graded) modal logic:

Theorem

A graded modal formula $\varphi \in \mathrm{ML}_k(\#)$ *is preserved under homomorphisms between pointed Kripke structures iff it is equivalent to an existential positive modal formula* $\psi \in \exists^+\mathrm{ML}_k$*.*

“Forcing” the bisimilar companion property

Tame vs wild

- Tame (bisimilar companion property): guarded and modal logics.
- Wild (no bisimilar companion property): bounded quantifier rank, finite variable logics, hybrid logic, etc.

Rossman's equirank homomorphism preservation theorem essentially forces the bisimilar companion property:

$$a \quad \leftrightarrow_k^{\mathcal{C}} \quad G_k a \longrightarrow b \qquad \text{vs} \qquad \begin{array}{ccccc} a^* & \leftrightarrow_k^{\mathcal{C}} & (G_k a)^* & & \\ \uparrow\downarrow & & \uparrow\downarrow & & \\ a & \rightleftarrows_k^{\mathcal{C}} & G_k a & \longrightarrow & b \end{array}$$

a^* and $(G_k a)^*$ are k-extendable covers of a and $G_k a$, respectively.

Axioms for the extensional category

We require that the category $\mathcal{E}$ have the following properties:

(E1) $\mathcal{E}$ has all finite limits and small colimits.

(E2) $\mathcal{E}$ is equipped with a proper factorisation system such that:

- Embeddings are stable under pushouts along embeddings.
- Pushout squares of embeddings are also pullbacks.
- Pushout squares of embeddings are stable under pullbacks along embeddings.

Remark

These are essentially the axioms for adhesive categories.

Example

$\mathcal{R}(\sigma)$ satisfies (E1)–(E2). Pointed σ-structures also satisfy (E1)–(E2). In fact, these axioms are stable under coslices.

Axioms for the resource-indexed adjunctions

We now assume that the extensional category $\mathcal{E}$ satisfies (E1)–(E2), and introduce conditions on the resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$. We require the following properties for all $k > 0$:

(A1) $\mathcal{C}^k_p$ is locally finite and has finitely many objects up to isomorphism.

(A2) For all paths $P \in \mathcal{C}^k_p$, $L_k P$ is finitely presentable in $\mathcal{E}$.

(A3) The path restriction property is satisfied.

(A4) An arrow $m \colon P \to R_k a$, with $P \in \mathcal{C}^k_p$, is an embedding in $\mathcal{C}_k$ precisely when $m^\# \colon L_k P \to a$ is an embedding in $\mathcal{E}$.

Theorem

Consider a resource-indexed arboreal adjunction between $\mathcal{E}$ and $\mathcal{C}$ satisfying (E1)–(E2) *and* (A1)–(A4). *For all $a \in \mathcal{E}$ and all $k > 0$, there exists a k-extendable cover of a.*

Corollary

(HP) *holds for all resource-indexed arboreal adjunctions satisfying* (E1)–(E2) *and* (A1)–(A4).

Applying the corollary to the Ehrenfeucht-Fraïssé resource-indexed arboreal adjunction, we recover the equirank HPT.

Relativisations to full subcategories are also available.

Remark

The hybrid comonads $\mathbb{H}_k$ do not seem to have the path restriction property, i.e. (A3) fails. Also, $\mathbb{P}_{n,k}$ does not satisfy (A4).

Thank you for your attention!