Linear arboreal categories

Samson Abramsky, Yoàv Montacute, Nihil Shah

University of Oxford

September 23, 2022

Spoiler-Duplicator game comonads originate from arboreal covers over a category of relational structures [AR21]

Comonads $\mathbb{C} = LR$ arising from a comonadic adjunction:

$$\mathcal{C} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}$$

$\mathcal{C}$ is category of 'tree'-shaped objects which describe a process that builds objects in extensional category $\mathcal{E}$

Arboreal categories admit a notion of bisimulation.

In the case of $\mathcal{E} = \mathbf{Struct}(\sigma)$ and game comonad $\mathbb{C}$, bisimulation in $\mathbf{EM}(\mathbb{C})$ captures the full fragment of the logic associated to the game comonad $\mathbb{C}$.

Arboreal category [AR21] axioms

If $\mathcal{C}$ is a path category, then :

- Has a subcategory of path objects $\mathcal{C}_p$
- $\mathcal{C}$ has all small coproducts of path objects
- Has notion of embedding $X \rightarrowtail Y$
- Every path object $P \in \mathcal{C}_p$ is connected, i.e. for all non-empty families of paths $\{P_i\}$, a morphism $P \to \coprod_i P_i$ factors as:

$$P \to P_j \to \coprod_i P_i$$

  for some $P_j \in \{P_i\}$.

$\mathcal{C}$ is a arboreal category if:

- Every object $X \in \mathcal{C}$ is a colimit of its path embeddings $P \rightarrowtail X$

There is a category of k-pebble forest covers of σ-structures $R^P_k(\sigma)$ [ADW17].

Objects are $(\mathcal{A}, \leq, p\colon A \to \{1, \ldots, k\})$ where $\leq$ is a forest order

$$\downarrow a = \{a' \in A \mid a' \leq a\} \text{ is a chain}$$

and $\leq$ and p are compatible with the σ-structure on $\mathcal{A}$

Morphisms are σ-morphisms which preserve p and the covering relation $\prec$.

$\mathbb{P}_k = U_k \circ G_k$ results from arboreal cover $U_k \dashv G_k$ where U_k is forgetful functor, $G_k\colon R(\sigma) \to R^P_k(\sigma)$ with $G_k(\mathcal{A}) = (\mathbb{P}_k\mathcal{A}, \sqsubseteq, \pi_A)$

This a comonadic adjunction $\mathbf{EM}(\mathbb{P}_k) \cong R^P_k(\sigma)$

There is a category of k-pebble linear forest covers of σ-structures $R_k^{PL}(\sigma)$.

Objects are $(\mathcal{A}, \leq, p\colon A \to \{1, \ldots, k\})$ where $\leq$ is a forest order

$$\downarrow a = \{a' \in A \mid a' \leq a\} \text{ is a chain}$$

$$\uparrow a = \{a' \in A \mid a \leq a'\} \text{ is a chain}$$

and $\leq$ and p are compatible with the σ-structure on $\mathcal{A}$

Morphisms are σ-morphisms which preserve p and the covering relation $\prec$.

$\mathbb{PR}_k = U_k \circ G_k$ results from arboreal cover $U_k \dashv G_k$ where U_k is forgetful functor, $G_k\colon R(\sigma) \to R_k^P(\sigma)$ with $G_k(\mathcal{A}) = (\mathbb{PR}_k\mathcal{A}, \sqsubseteq, \pi_A)$

This a comonadic adjunction $\mathbf{EM}(\mathbb{PR}_k) \cong R_k^{PL}(\sigma)$

$\mathbb{PR}_k$ [MS21] is a 'linear' variant of the pebbling $\mathbb{P}_k$ comonad, e.g.

- $\mathcal{A} \to \mathbb{P}_k\mathcal{A} \Leftrightarrow$ tree decomposition of width $< k$.
- $\mathcal{A} \to \mathbb{PR}_k\mathcal{A} \Leftrightarrow$ path decomposition of width $< k$
- $\mathbb{PR}_k$ captures the restricted conjunction fragments of the logics captured by the Kleisli category of $\mathbb{P}_k$

$\mathbb{PR}_k$ is an arboreal cover, it seems to have additional 'linear' structure when comparing $\mathbb{P}_k$

Linear structure means that $\mathbb{PR}_k$ is utilising some notion of trace equivalence/inclusion

- e.g. all-in-one k-pebble game. Spoiler announces his full play in the k-pebble game. Duplicator responds with a full play

How to strengthen the arboreal category axioms to capture the linear behavior of $\mathbf{EM}(\mathbb{PR}_k)$, but exclude the branching behavior of $\mathbf{EM}(\mathbb{P}_k)$?

Is there an abstract way for defining the 'linear' variant for any arboreal cover?

The adjunction yielding the modal comonad $\mathbb{M}$ is the most 'tame' example of an arboreal cover.

Construct a linear variant of $\mathbb{M}^L$ of the modal comonad $\mathbb{M}$ capturing trace equivalence

Extensional category

Relational signature σ with unary relations $\{P_l\}_{l \in \mathsf{AP}}$ and binary transition relations $\{R_\alpha\}_{\alpha \in \mathsf{Act}}$

Category of pointed σ-structures $\mathbf{Struct}_\star(\sigma)$

- Objects $(\mathcal{A}, a_0)$ are universes $U(\mathcal{A}, a_0) = A$, $P_l^{\mathcal{A}} \subseteq A$, $R_\alpha^{\mathcal{A}} \subseteq A^2$, and $a_0 \in A$
- Morphisms are set functions that preserve the relations and the distinguished point

We define a 'linear' variant $\mathbb{M}^L$ of $\mathbb{M}$

Given a pointed σ-structure $(\mathcal{A}, a_0)$ with $\sigma = \{P_l\}_{AP} \cup \{R_\alpha\}_{Act}$, we define $L\colon A \to \mathcal{P}(AP)$

$$L(a) = \{l \mid a \in P_l^{\mathcal{A}}\}$$

$$\text{runs}_n(\mathcal{A}, a_0) = \{[a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} a_{n-1} \xrightarrow{\alpha_n} a_n] \mid R_{\alpha_i}^{\mathcal{A}}(a_{i-1}, a_i)\}$$

$\mathbf{Trace}_n(a_0)$ is the set of all sequences
$L(a_0)\alpha_1 L(a_1) \dots L(a_{n-1})\alpha_n L(a_n)$
such that $[a_0 \xrightarrow{\alpha_1} a_1 \dots a_{n-1} \xrightarrow{\alpha_n} a_n] \in \text{runs}_n(\mathcal{A}, a_0)$

$$\mathbf{Trace}(a_0) = \bigcup_{n \in \omega} \mathbf{Trace}_n(a_0)$$

- $(\mathcal{A}, a_0) \subseteq_{tr} (\mathcal{B}, b_0)$ if
    1. For all $[a_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} a_n] \in \text{runs}_n(\mathcal{A}, a_0)$, there exists a $[b_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} b_n] \in \text{runs}_n(b_0)$ such that $L(a_i) \subseteq L(b_i)$
- $(\mathcal{A}, a_0) \subseteq_{ptr} (\mathcal{B}, b_0)$ if $\textbf{Trace}(a_0) \subseteq \textbf{Trace}(b_0)$

$R^{ML}(\sigma)$ consists of Kripke structures $(\mathcal{A}, a_0, \leq)$ which are trees such that:

(T) $a \prec a'$ iff there exists a unique $\alpha \in \mathrm{Act}$, $R_\alpha(a, a')$

(L) If $a \neq a_0$, then there is at most one a' such that $a \prec a'$.

Paths in $R^{ML}(\sigma)$ are those structures where (L) also holds for a_0.

$G\colon \mathbf{Struct}_\star(\sigma) \to R^{ML}(\sigma)$ maps $(\mathcal{A}, a_0)$ to $(U(G(\mathcal{A})), a_0, \leq)$ such that

- Universe of $U(G(\mathcal{A}))$ consists of a_0 and pairs (s, i) where $s \in \mathrm{runs}_n(\mathcal{A}, a_0)$ and $i \in \{1, \ldots, n\}$
- $R_\alpha^{U(G(\mathcal{A}))}$ has pair of pairs (s, i) and $(s, i+1)$ if the i-th transition of s is α
- $R_\alpha^{U(G(\mathcal{A}))}$ has pair a_0 and $(s, 1)$ if the first transition of s is α
- (s, i) has label P iff i-th state of s has label P
- $a_0 \leq (s, 1)$ or $(s, i) \leq (s, j)$ iff $i \leq j$

$\mathbb{M}^L = U \circ G$ results from arboreal cover $U \dashv G$ of $R^{ML}(\sigma)$ over $\mathbf{Struct}_\star(\sigma)$

Proposition

The following are equivalent:

- *There exists a coalgebra morphism* $G(\mathcal{A}, a_0) \to (\mathcal{B}, b_0)$
- $(\mathcal{A}, a) \subseteq_{tr} (\mathcal{B}, b_0)$

A pathwise embedding $f\colon X \to Y$, if given a path embedding $e\colon P \rightarrowtail X$, $f \circ e\colon P \rightarrowtail Y$

Proposition

The following are equivalent

- *Pathwise embedding* $G(\mathcal{A}, a_0) \to G(\mathcal{B}, b_0)$
- $(\mathcal{A}, a_0) \subseteq_{ptr} (\mathcal{B}, b_0)$

$\mathbb{PR}_k$ linear variant of $\mathbb{P}_k$, $\mathbb{M}^L$ linear variant of $\mathbb{M}$

Need to strengthen the arboreal category axioms to include $\mathbb{PR}_k$,$\mathbb{M}^L$, but exclude 'branching' $\mathbb{P}_k, \mathbb{M}$

Plays are full paths rather then just the next move

Maximal path embeddings

Working in the setting of a path category $\mathcal{C}$

Definition

Given a path embedding $P \rightarrowtail \mathcal{A}$ if for all commuting diagrams of the form:

$$\begin{array}{ccc} P & \overset{i}{\rightarrowtail} & P' \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{1}$$

we have that i is an isomorphism, then $P \rightarrowtail X$ is a *maximal path embedding*.

Equivalently, the maximal path embeddings $m\colon P \rightarrowtail X$ are in the top elements $[m]$ of $\mathbf{Paths}(X)$

Linear arboreal categories

$\mathcal{C}$ is an arboreal category if:

- $\mathcal{C}$ is a path category
- Every object $X \in \mathcal{C}$ is a colimit of its path embeddings $P \rightarrowtail X$
  - equivalently, $J \colon \mathcal{C}_p \hookrightarrow \mathcal{C}$ is dense.

$\Leftarrow$

$\mathcal{C}$ is an linear arboreal category if:

- $\mathcal{C}$ is a path category
- Every object $X \in \mathcal{C}$ is a coproduct of its maximal path embeddings $P \rightarrowtail X$
  - equivalently, $J \colon \mathcal{C}_p \hookrightarrow \mathcal{C}$ is discretely dense.

Deriving linear variant

Suppose we have a arboreal category $\mathcal{C}$ with subcategory of paths $\mathcal{C}_p$.

Let $\mathcal{C}^L$ be a the small coproduct completion of $\mathcal{C}_p$

By $\mathcal{C}$ closed under coproducts, there is an inclusion $J\colon \mathcal{C}^L \hookrightarrow \mathcal{C}$.

J has a right adjoint $T\colon \mathcal{C} \to \mathcal{C}^L$, sending an object to its coproduct of maximal paths.

$\mathcal{C}^L$ is a coreflective subcategory of $\mathcal{C}$

$$\mathcal{C}^L \underset{T}{\overset{J}{\underset{\bot}{\rightleftarrows}}} \mathcal{C} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}$$

Given an arboreal cover $L \dashv R$ of $\mathcal{C}$ over $\mathcal{E}$ yielding comonad $\mathbb{C} = LR$, we obtain a arboreal cover $LJ \dashv TR$ of $\mathcal{C}^L$ over $\mathcal{E}$ yielding $\mathbb{C}^L = LJTR$

- e.g. obtaining $\mathbb{PR}_k$ from $\mathbb{P}_k$ and $\mathbb{M}^L$ from $\mathbb{M}$.

If ε is counit of $J \dashv T$, then $L(\varepsilon_R)\colon \mathbb{C}^L \to \mathbb{C}$ is a comonad morphism.

- e.g. $\mathbb{PR}_k \to \mathbb{P}_k$ and $\mathbb{M}^L \to \mathbb{M}$

Conclusion

Suggests that trace inclusion and equivalence are related to restricted conjunction fragments in modal logic.

Potentially, a framework to prove various preservation theorems about trace inclusion/equivalence.

More generally, obtain preservation theorems for restriction conjunction fragments of logics "for free".

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