On three games of logic

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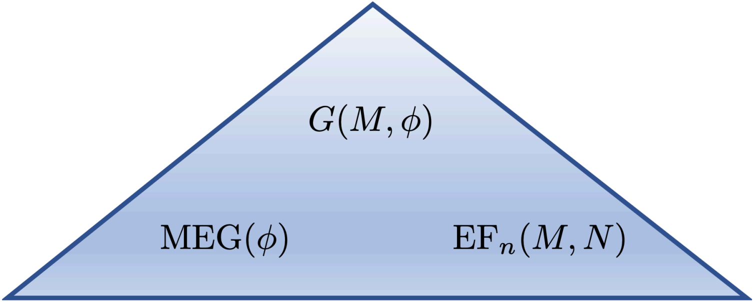
The three games

1. Evaluation Game: "ϕ is true in M?"
2. Model Existence Game: "ϕ is consistent?"
3. EF (Ehrenfeucht-Fraïssé) game: "some sentence separates M from N?"

Really just one game. Essential to logic. Distinguishes logic from algebra, topology, analysis, etc.

The strategic balance of logic



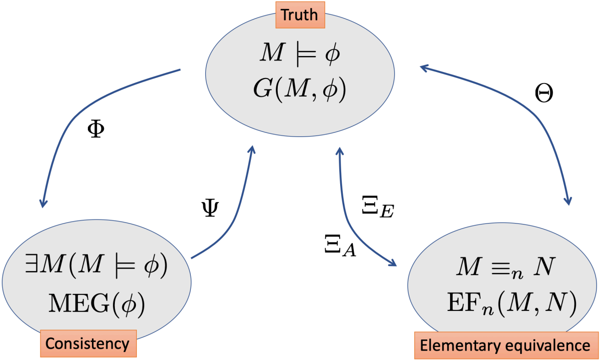
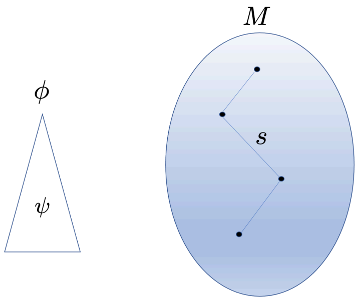


Figure: The translations of strategies.

1st game: Evaluation (a.k.a. semantic) Game $G(M, \phi)$

- Two players Abelard and Eloise.
- M a model, ϕ a sentence of $L_{\infty\omega}$.
- s an assignment.
- Pairs (ψ, s) are positions.
- A token: At any time, one of the players has the token and the other player is called "the opponent".
- Starting position is $(\phi, \emptyset)$ and Eloise has the token.



- Intuitively, the one with the token defends during the game $G(M, \phi)$ the proposition that ϕ is (informally) true in M under the assignment s, and the opponent doubts it.

The **rules** in position (ψ, s) are: If ψ is

(1) atomic, the game ends and the one with the token wins if s satisfies ψ in M. Otherwise the opponent wins.

(2) $\bigvee_{i \in I} \psi_i$, the one with the token chooses $i \in I$ and the next position is (ψ_i, s).

(3) $\bigwedge_{i \in I} \psi_i$, the opponent chooses $i \in I$ and the next position is (ψ_i, s).

(4) $\exists x \theta$, the one with the token chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

(5) $\forall x \theta$, the opponent chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

(6) $\neg \theta$, the token is passed to the opponent and the next position is (θ, s).

- We say that ϕ is true in M if Eloise has a winning strategy in $G(M, \phi)$.
- This is the game-theoretical meaning of truth in a model.
- We can go further and say that the *game* $G(M, \phi)$ **is** the meaning of ϕ in M. Here meaning would be a broader concept than the mere truth or falsity of ϕ.
- [Wittgenstein, 1953], [Henkin, 1961], [Hintikka, 1968]

- The game $G(M, \bigwedge_{i \in I} \psi_i)$ is intimately related to the games $G(M, \psi_i)$, $i \in I$.
- The same with $G(M, \bigvee_{i \in I} \psi_i)$, $G(M, \exists x \phi)$ and $G(M, \forall x \phi)$.
- This phenomenon is a manifestation of the broader concept of *compositionality*.
- The games $G(M \times N, \phi)$, $G(M + N, \phi)$, and $G(\Pi_i M_i / F, \phi)$ are intimately related to the games $G(M, \phi)$, $G(N, \phi)$ and $G(M_i, \phi)$ [Feferman, 1972].

- If ϕ is **propositional** i.e. has only zero-place relation symbols, no constant or function symbols, and no quantifiers, then only moves (1)-(3) occur in $G(M, \phi)$, and the assignments can be forgotten.
- If ϕ is **positive**, the game $G(M, \phi)$ has no moves of type (6).
- If ϕ is **universal**, the game $G(M, \phi)$ has no moves of type (4).
- If it is **existential**, the game has no moves of type (5).
- If **universal-existential**, then all type (5) moves come before type (4) moves.
- If we add **new logical operations** to our logic, such as generalized quantifiers or higher order quantifiers, it is clear how to modify the game $G(M, \phi)$ to accommodate the new logical operations.
- If M is a Kripke-model and ϕ a sentence of modal logic, the game $G(M, \phi)$ is entirely similar.

2nd game: Model Existence Game MEG(ϕ)

- We have a sentence and we ask whether the sentence has a model. Thus this is about *consistency* and its opposite, *contradiction*.
- Is there some model M such that Eloise can win $G(M, \phi)$?
- Suppose ϕ is a sentence of $L_{\omega_1\omega}$. Logical operations: $\neg, \wedge_n, \vee_n, \forall$ and $\exists$.
- We assume that ϕ is in NNF (Negation Normal Form, negation only in front of atomic formulas).

- The game MEG(ϕ) has two players Abelard and Eloise.
- Intuitively, Eloise defends the proposition that ϕ has a model and Abelard doubts it. Abelard expresses his doubt by asking questions.
- We let $C = \{c_0, c_1, \ldots, c_n, \ldots\}$ be a set of new distinct constant symbols. Intuitively these are names of elements of the supposed model.
- A C-assignment is an assignment with values in C.

A position is a finite set S of pairs (ψ, s), where s is an assignment into C. Starting position is $\{(\phi, \emptyset)\}$. Abelard chooses a pair $(\psi, s) \in S$.

(1) $(\bigwedge_n \psi_n, s)$: Next position is $S \cup \{(\psi_n, s)\}$ for some n, and Abelard decides which.

(2) $(\bigvee_n \psi_n, s)$: Next position is $S \cup \{(\psi_n, s)\}$ for some n, and Eloise decides which.

(3) $(\forall x\theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$, and Abelard chooses $c \in C$.

(4) $(\exists x\theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$, and Eloise chooses $c \in C$.

If $(\psi, s), (\neg\psi, s') \in S$ for atomic ψ, where $s(x) = s'(x)$ for all x in ψ, Abelard wins.

- Gentzen's natural deduction,
- [Beth, 1955],
- [Hintikka, 1955],
- [Smullyan, 1963],
- [Makkai, 1969].
- Craig Interpolation Theorem.
- Completeness Theorem.
- Preservations Theorems.

Truth $\Rightarrow$ consistency

Theorem

Suppose $\phi \in L_{\omega_1\omega}$ is a sentence in NNF. Every strategy τ of Eloise in $\boldsymbol{G(M,\phi)}$ determines a strategy $J_E(\tau)$ of Eloise in MEG(ϕ). If τ is a winning strategy, then so is $J_E(\tau)$.

(We assume the vocabulary of M is countable.)

- There is a countable submodel N of M such that τ is a strategy of Eloise in $G(N, \phi)$. Let $\pi : C \to N$ be an onto map.
- A pair (ψ, s) is a τ-position if there is there is some sequence of positions in $G(N, \phi)$, following the rules of $G(N, \phi)$ starting with $(\phi, \emptyset)$, Eloise using τ, which ends at (ψ, s).
- A C-translation of the τ-position (ψ, s) is a pair (ψ, s') where s' is a C-assignment with $\pi(s'(x)) = s(x)$ for all x.
- The strategy $J_E(\tau)$ of Eloise in MEG(ϕ) is to make sure that at all times the position S consists only of C-translations of τ-positions.

$$
\begin{array}{ccccc}
 & C & \text{——} & N \subseteq & M & \\
J_E(\tau) & \Big| & & & \Big| & \tau \\
 & \phi & & & \phi &
\end{array}
$$

Figure: From model to model existence.

Consistency ⇒ model and truth in it

Theorem

Suppose $\phi \in L_{\omega_1\omega}$ is a sentence in NNF. Every strategy τ of Eloise in MEG(ϕ) determines a model $M(\tau)$ and a strategy $J_E(\tau)$ of Eloise in $\boldsymbol{G(M(\tau), \phi)}$. If τ is winning, then so is $J_E(\tau)$.

[Beth, 1955]

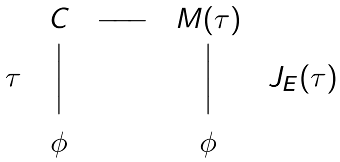


Figure: From model existence to a model.

Let σ_0 be the following enumeration strategy of Abelard in MEG(ϕ): During the game Abelard makes sure that if S is the position, then:

1. If $(\bigwedge_n \psi_n, s) \in S$, then during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\psi_0, s)\}$ and at some further position $S'' \supseteq S'$ he will decide that the next position is $S'' \cup \{(\psi_1, s)\}$, etc.
2. If $(\bigvee_n \psi_n, s) \in S$, then at some position $S' \supseteq S$ Abelard asks Eloise to choose whether the next position is $S' \cup \{(\psi_0, s)\}$ or $S' \cup \{(\psi_1, s)\}$ or ...
3. If $(\forall x\theta, s) \in S$, then for all n during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\theta, s(c_n/x)\}$.
4. If $(\exists x\theta, s) \in S$, then at some position $S' \supseteq S$ Abelard will ask Eloise to choose n after which the next position is $S' \cup \{(\theta, s(c_n/x))\}$.

- Let us play MEG(ϕ) while Abelard uses this strategy and Eloise plays τ.
- Let $\mathcal{S} = \langle S_n : n < \omega \rangle$ be the infinite sequence of positions during this play. Note that $S_n \subseteq S_{n+1}$ for all n. Let $\mathbf{\Gamma}$ be the union of all the positions in $\mathcal{S}$.
- We build a model $M = M(\tau)$ as follows[1]: The domain of the model is $\{c_n : n \in \mathbb{N}\}$. If R is a relation symbol, then we let $R(c_{n_0}, \ldots, c_{n_k})$ hold in M if $(R(x_{n_0}, \ldots, x_{n_k}), s) \in \mathbf{\Gamma}$ for some s such that $s(x_i) = c_i$ for $i = n_0, \ldots, n_k$.
- The strategy $\boldsymbol{J_E(\tau)}$ of Eloise in $G(M, \phi)$ is the following: She makes sure that if the position in $G(M, \phi)$ is (ψ, s), then $(\psi, s) \in \mathbf{\Gamma}$. Let us see that she can follow the strategy throughout the game:

[1] We assume, for simplicity, that ϕ has a relational vocabulary and does not contain the identity symbol.

- If ψ is $\exists x\theta$, then Eloise should choose for which n the next position is $(\theta, s(c_n/x))$.
- We know $(\exists x\theta, s) \in S$ for some position S during the game, because $(\exists x\theta, s) \in \mathbf{\Gamma}$.
- By how σ_0 was defined, Abelard has at some later position $S' \supseteq S$ asked Eloise to choose n for which the next position would be $S' \cup \{(\theta, s(c_n/x))\}$.
- The strategy τ has directed Eloise to indeed choose an n leading to the new position $S' \cup \{(\theta, s(c_n/x))\}$.
- Thus $(\theta, s(c_n/x)) \in \mathbf{\Gamma}$ and she can safely play $(\theta, s(c_n/x))$ in $G(M, \phi)$.

- A winning strategy of Eloise in MEG(ϕ) can be conveniently given in the form of a so-called *consistency property*, which is just a set of finite sets of sentences satisfying conditions which essentially code a winning strategy for Eloise in MEG(ϕ).
- Sometimes it is more convenient to use a consistency property than Model Existence Game. But as far as strategies of Eloise are concerned, the two are one and the same thing.
- Consistency properties have been successfully used to prove interpolation and preservations results in model theory, especially infinitary model theory [Makkai, 1969].

- Suppose now Abelard has a winning strategy in MEG(ϕ).
- We can form a tree, a Beth Tableaux, of all the positions when Abelard plays his winning strategy and we stop playing as soon as Abelard has won.
- Every branch of the tree is finite and ends in a position which includes a contradiction.
- We can then view this tree as a proof of $\neg\phi$. In this sense the Model Existence Game builds a bridge between proof theory and model theory.
- Strategies of Abelard direct us to proof theory, while strategies of Eloise direct us to model theory.

Apart from first order and infinitary logic, the Model Existence Game can be used in the proof theory and model theory of

- propositional and modal logic.
- logic with generalized quantifiers and higher order logic.
- weak models, which have to be transformed to real models by a model theoretic argument [Keisler, 1970].
- general models for higher order logics [Henkin, 1950].
- infinitary logic $L_{\kappa\lambda}$,
- chain models, rather than real models.

3rd game: EF (Ehrenfeucht-Fraïssé) game

- In the EF game we have a model (actually two models) but no sentence.
- The sentence should be true in one but false in the other.
- In the EF game strategies of Eloise track possibilities for elementary equivalence and the strategies of Abelard track possibilities for a separating sentence.
- [Fraïssé, 1954], [Ehrenfeucht, 1961]
- M and N are two structures for the same vocabulary L.

Definition

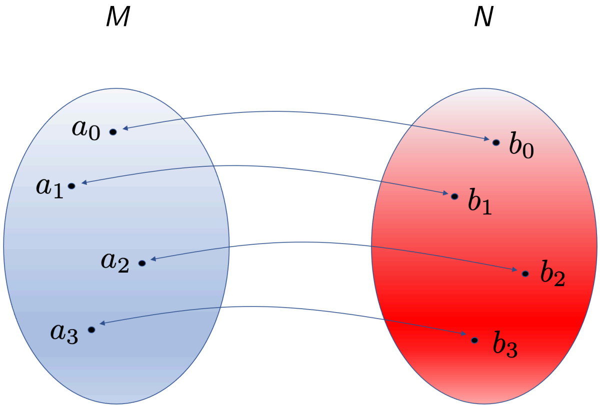
The game $\mathrm{EF}_m(M, N)$ has two players Abelard and Eloise and m moves. A position is a set

$$s = \{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\} \tag{1}$$

of pairs of elements such that the a_i are from M and the b_i are from N, and $n \leq m$. In the beginning the position is $\emptyset$. The rules:

1. Abelard may choose some $a_n \in M$. Then Eloise chooses $b_n \in N$ and the next position is $s \cup \{(a_n, b_n)\}$.
2. Abelard may choose some $b_n \in N$. Then Eloise chooses $a_n \in M$ and the next position is $s \cup \{(a_n, b_n)\}$.

Abelard wins if during the game the position (2) is such that $(a_0, \ldots, a_{n-1})$ satisfies some literal in M but $(b_0, \ldots, b_{n-1})$ does not satisfy the corresponding literal in N.


M
N
a_0
a_1
a_2
a_3
b_0
b_1
b_2
b_3

- Intuitively, Eloise defends the proposition that M and N are very similar.
- Abelard doubts this similarity.
- If Eloise knows an isomorphism $f : M \to N$ she can respond by playing always so that $b_n = f(a_n)$.
- Two models of (any) size $\geq n$ in the empty vocabulary.
- Two finite linear orders of (any) size $\geq 2^n$.
- This game is **determined**.
- How long games can Eloise win in case $M \not\cong N$?
- A logician's version of isomorphism.
- A formula "is" this game.

- Before going into proofs, we generalize the game a little: Dynamic EF-game.
- The length of the game is determined dynamically, move by move.
- There is an ordinal-clock.

The game $\mathbf{EFD}_\beta(M,N)$ has two players Abelard and Eloise and an apriori unknown finite number of moves. A position is a set

$$s = \{(\alpha_0, a_0, b_0), \ldots, (\alpha_{n-1}, a_{n-1}, b_{n-1})\} \tag{2}$$

of triples of elements such that $\beta > \alpha_0 > \alpha_1 > \ldots$, the a_i are from M and the b_i are from N, and $n < \omega$. In the beginning the position is $\emptyset$. The rules:

1. Abelard chooses $\alpha_n < \alpha_{n-1}$, if he can ($\alpha_{-1} = \beta$).
2. Abelard may choose some $a_n \in M$. Then Eloise chooses $b_n \in N$ and the next position is $s \cup \{(\alpha_n, a_n, b_n)\}$.
3. Abelard may choose some $b_n \in N$. Then Eloise chooses $a_n \in M$ and the next position is $s \cup \{(\alpha_n, a_n, b_n)\}$.

Abelard wins if during the game the position (2) is such that $(a_0, \ldots, a_{n-1})$ satisfies some literal in M but $(b_0, \ldots, b_{n-1})$ does not satisfy the corresponding literal in N.

- Note that EFD_n is the same game as EF_n.
- How long dynamic games can Eloise win when $M \not\cong N$?
- Interesting also for transfinite games, but then we use trees as clocks.

Quantifier rank in $L_{\infty\omega}$

$$
\begin{cases}
qr(\phi) & = & 0, \text{ if } \phi \text{ atomic} \\
qr(\neg\phi) & = & qr(\phi) \\
qr(\forall x\phi) & = & qr(\phi) + 1 \\
qr(\exists x\phi) & = & qr(\phi) + 1 \\
qr(\bigwedge_{i\in I} \phi_i) & = & \sup\{qr(\phi_i) : i \in I\} \\
qr(\bigvee_{i\in I} \phi_i) & = & \sup\{qr(\phi_i) : i \in I\}
\end{cases}
$$

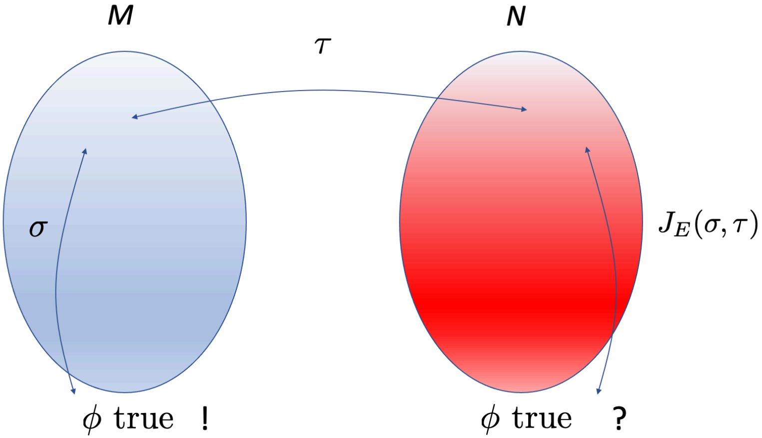
Strategy of Eloise $\Rightarrow$ elementary equivalence

Transfer of truth

Theorem

Suppose ϕ is an $L_{\infty\omega}$-sentence of quantifier rank $\leq \beta$. Every strategy τ of Eloise in $\mathrm{EFD}_\beta(M, N)$, and every strategy σ of Eloise in $G(M, \phi)$ determine a strategy $J_E(\sigma, \tau)$ of Eloise in $G(N, \phi)$. If τ and σ are winning strategies, then so is $J_E(\sigma, \tau)$.

[Ehrenfeucht, 1961]


M
N
τ
σ
$J_E(\sigma, \tau)$
ϕ true !
ϕ true ?

- We call a position of the game $\mathrm{EFD}_\beta(M, N)$ a **τ-position** if it arises while Eloise is playing τ.
- We call a position of the game $G(M, \phi)$ a **σ-position**, if it arises while Eloise is playing σ.
- During the game $G(N, \phi)$ Abelard and Eloise choose some elements $b_0, \ldots, b_{n-1}$ of N. We submit these elements to the game $\mathrm{EF}_m(M, N)$ and use τ to get to the other side, to model M, and obtain elements $a_0, \ldots, a_{n-1}$ of M. There we use these elements as corresponding moves of Abelard and Eloise in the game $G(M, \phi)$, Eloise playing her winning strategy σ. Thus we play two games in synchrony. This is how it happens:

- If the position of the game $G(N, \phi)$ is (ψ, s), the **strategy** $J_E(\sigma, \tau)$ of Eloise is to play simultaneously $G(N, \phi)$, $\mathrm{EFD}_\beta(M, N)$, and $G(M, \phi)$, as well as make sure that if

$$\pi = \{(\alpha_0, a_0, b_0), \ldots, (\alpha_{n-1}, a_{n-1}, b_{n-1})\}$$

is the current τ-position in $\mathrm{EFD}_\beta(M, N)$, then $qr(\psi) = \alpha_{n-1}$, and if we denote

$$s(x) = \pi(s'(x)))$$

for all x in the domain of s, then (ψ, s') is the current σ-position in $G(M, \phi)$.

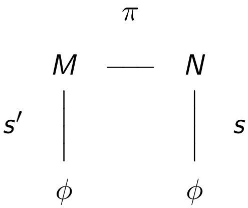


Figure: The strategy $J_E(\sigma, \tau)$, Let us check that it is possible for Eloise to play this strategy: If the position in $G(N, \phi)$ is (ψ, s) where ψ is:

1. A literal, the game ends.
2. $\bigwedge_i \phi_i$. The opponent chooses i and the next position is (ψ_i, s). Whichever (s)he chooses, we let the opponent make the respective move (ψ_i, s') in $G(M, \phi)$.
3. $\bigvee_i \phi_i$. The player with the token chooses i and the next position is (ψ_i, s). Whichever (s)he chooses, we let the player with the token make the respective move (ψ_i, s') in $G(M, \phi)$.

4. $(\forall x\theta, s)$. The opponent chooses $b_n \in N$ and the next position is (θ, t), $t = s(b_n/x)$. We continue the game $\mathrm{EF}_m(M, N)$ from the τ-position $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ letting Abelard play $b_n \in N$. The strategy τ tells Eloise to choose $a_n \in M$ so that

$$\pi' = \{(a_0, b_0), \ldots, (a_n, b_n)\} \tag{3}$$

is again a τ-position. Now we continue the game $G(M, \phi)$ from position $(\forall x\theta, s')$ by letting the opponent play a_n. We reach the position (θ, t'), $t' = s'(a_n/x)$, which is still a σ-position, and we have $t'(y) = \pi'(t(y))$ for all y in the domain of t'.

5. $(\exists x\theta, s)$. Now we continue the game $G(M, \phi)$ from position $(\exists x\theta, s')$ by letting the player with the token play, according to σ, an element a_n and we reach a new σ-position (θ, t'), $t' = s'(a_n/x)$. We continue the game $\mathrm{EF}_m(M, N)$ from the τ-position $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ letting Abelard play $a_n \in M$. The strategy τ tells Eloise to choose $b_n \in N$ so that (4) is again a τ-position. We reach the position (θ, t), $t = s(b_n/x)$, and we have $t(y) = \pi(t'(y))$ for all y in the domain of t.
   In both 4. and 5. it is the token that decides whether Abelard plays in M or in N.
6. $(\neg\theta, s)$. In both $G(M, \phi)$ and $G(N, \phi)$ the token is passed to the other player. The next position in $G(M, \phi)$ is (θ, s') and in $G(M, \phi)$ it is (θ, s).
   Thus negation switches whether Abelard plays in M or in N.

6. If σ is a winning strategy and the game $G(N, \phi)$ ends in the position (ψ, s), where ψ is a literal, then the one with the token wins because then (ψ, s') is a σ-position meaning that s' satisfies the literal ψ in M, and τ being a winning strategy this means that s satisfies the literal ψ in N.

- There is a tight connection between σ, τ and $J_E(\sigma, \tau)$. This is reflected in a connection between ϕ and $\mathrm{EFD}_\beta(M, N)$.
- If the non-logical symbols of ϕ are in $L' \subset L$, then it suffices that τ is a strategy of Eloise in the game $\mathrm{EFD}_\beta(M \restriction L', N \restriction L')$ between the reducts $M \restriction L'$ and $N \restriction L'$.
- If we know more about the syntax of ϕ, for example that it is existential, universal or positive, we can modify $\mathrm{EFD}_\beta(M, N)$ accordingly by stipulating that Abelard only moves in M, only moves in N, or that he has to win by finding an atomic (rather than literal) relation which holds in M but not in N.
- Winning strategies of Eloise for the EF game are a standard method for showing that certain kinds of sentences do not exist. E.g. countability, well-foundedness, etc

Strategies of Abelard $\approx$ separating sentences

From a separating sentence to a strategy of Abelard.

Theorem

Suppose M and N are models and β is an ordinal. Suppose ϕ is a sentence in $L_{\infty\omega}$ of quantifier rank $\leq \beta$.

1. *There is a mapping J_A such that if τ is a strategy of Eloise in $G(M, \phi)$ and σ is a strategy of Abelard in $G(N, \phi)$, then $J_A(\tau, \sigma)$ is a strategy of Abelard in $\mathrm{EFD}_\beta(M, N)$.*
2. *If τ and σ are winning strategies, then $J_A(\tau, \sigma)$ is a winning strategy.*

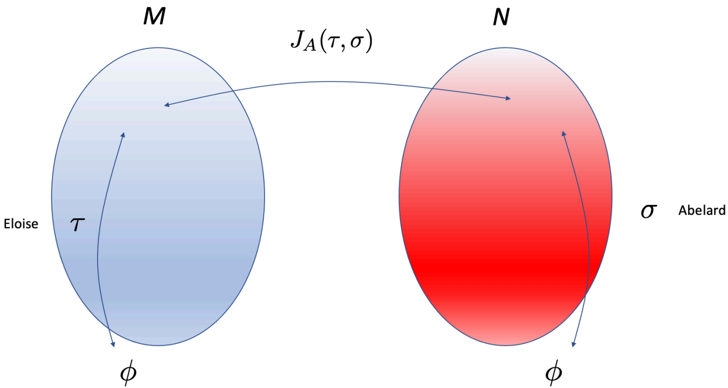
- We call a position of the game $G(M, \phi)$ a **τ-position** if it arises while Eloise is playing τ.
- We call a position of the game $G(N, \phi)$ a **σ-position**, if it arises while Abelard is playing σ.
- If the position of the game $\mathrm{EFD}_\beta(M, N)$ is

$$\pi = \{(\alpha_0, a_0, b_0), \ldots, (\alpha_{n-1}, a_{n-1}, b_{n-1})\},$$

the **strategy** $J_A(\sigma, \tau)$ of Abelard is to make sure that if the current τ-position in $G(M, \phi)$ is (ψ', s') and the current σ-position in $G(N, \phi)$ is (ψ, s), then $\psi = \psi'$, $\alpha_{n-1} = qr(\psi)$, and

$$s(x) = \pi(s'(x)))$$

for all x in the domain of s.



M
N
$J_A(\tau, \sigma)$
Eloise
τ
σ
Abelard
ϕ
ϕ

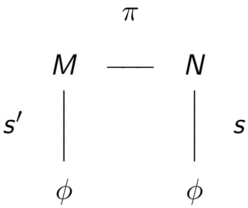


Figure: The strategy $J_A(\tau, \sigma)$, Let us check that it is possible for Eloise to play this strategy:

1. If the position is (ψ, s) where ψ is a literal or $\alpha_{m-1} = 0$, the game ends.
2. If the position is (ψ, s) where is ψ is $\bigwedge_i \phi_i$, then Abelard chooses i, and the next position is (ψ_i, s). Whichever he chooses, we let Abelard make the respective move (ψ_i, s') in $G(M, \phi)$.
3. If the position is (ψ, s) where is ψ is $\bigvee_i \phi_i$, then Eloise chooses i as follows. Since (ψ, s') is a σ-position, the strategy σ tells Eloise which of (ψ_i, s') to play in $G(M, \phi)$. Then Eloise plays the respective (ψ_i, s) in $G(N, \phi)$.

4. If the position is (ψ, s) is $(\forall x\theta, s)$, then Abelard chooses $\alpha_n < \alpha_{n-1}$ and $b_n \in N$ and the next position is (θ, t), $t = s(b_n/x)$. We continue the game $\mathrm{EFD}_\beta(M, N)$ from the τ-position $\{(\alpha_0, a_0, b_0), \ldots, (\alpha_{n-1}, a_{n-1}, b_{n-1})\}$ letting Abelard play b_n. Then Eloise chooses some $a_n \in M$ and

$$\pi' = \{(\alpha_0, a_0, b_0), \ldots, (\alpha_n, a_n, b_n)\} \tag{4}$$

is the next position. Now we continue the game $G(M, \phi)$ from position $(\forall x\theta, s')$ by letting Abelard play a_n. We reach the position (θ, t'), $t' = s'(a_n/x)$, which is still a σ-position, and we have $t'(y) = \pi'(t(y))$ for all y in the domain of t'.

5. The position is (ψ, s) is $(\exists x\theta, s)$. Now we continue the game $G(M, \phi)$ from position $(\exists x\theta, s')$ by letting Eloise play, according to σ, an element a_n and we reach a new σ-position (θ, t'), $t' = s'(a_n/x)$. We continue the game $\mathrm{EFD}_\beta(M, N)$ from the τ-position $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ letting Abelard play $\alpha_n = qr(\theta)$ and a_n. Then Eloise chooses some $b_n \in N$ and (4) is the next position. We reach the position (θ, t), $t = s(b_n/x)$, and we have $t(y) = \pi(t'(y))$ for all y in the domain of t.

6. If τ and σ are winning strateges, and the game $G(N, \phi)$ ends in the position (ψ, s), where ψ is a atomic, then Abelard wins the EF-game because

   Case 1: Eloise has the token in both games: Since (ψ, s') is a τ-position, s' satisfies ψ in M, and on the other hand, σ being a winning strategy of Abelard, s fails to satisfy ψ in N.

   Case 2: Abelard has the token in both games: Since (ψ, s') is a τ-position, s' does not satisfy ψ in M, and on the other hand, σ being a winning strategy of Abelard, s satisfies ψ in N.

Strategies of Abelard and separating sentences

From a strategy of Abelard to a separating sentence.

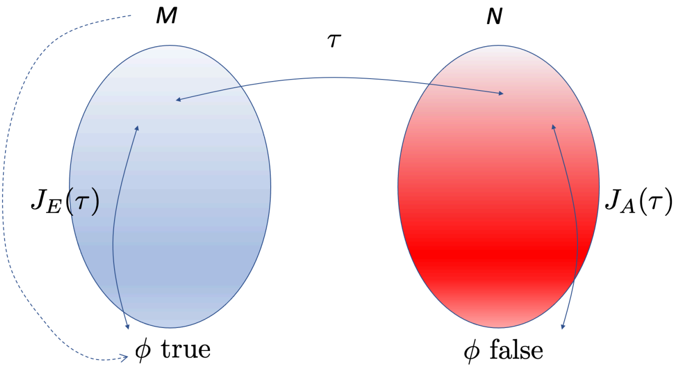
Theorem

Suppose M is a model and β is an ordinal. There is a sentence $\phi_M \in L_{\infty\omega}$ of quantifier rank $\leq \beta$ and a winning strategy $J_E(\tau)$ of Eloise in $G(M, \phi_M)$ such that the following hold: Suppose N is a model of the same vocabulary.

1. *There is a mapping J_A such that if τ is a strategy of Abelard in $\mathrm{EFD}_\beta(M, N)$, then $J_A(\tau)$ is a strategy of Abelard in $G(N, \phi_M)$.*
2. *If τ is a winning strategy, then $J_A(\tau)$ is a winning strategy.*

Note: If L is finite and relational, and β is finite, the sentence ϕ_M is logically equivalent to a first order sentence of quantifier rank $\leq \beta$.

[Ehrenfeucht, 1961]



M
N
τ
$J_E(\tau)$
$J_A(\tau)$
ϕ true
ϕ false

Suppose s is an assignment into M with domain $\{x_0, \ldots, x_{n-1}\}$.
Let

$$\begin{array}{rcl} \psi^{0,n}_{M,s} & = & \bigwedge_i \psi_i \\ \psi^{\xi+1,n}_{M,s} & = & (\forall x_n \bigvee_{a \in M} \psi^{\xi,n+1}_{M,s(a/x_n)}) \wedge (\bigwedge_{a \in M} \exists x_n \psi^{\xi,n+1}_{M,s(a/x_n)}) \\ \psi^{\nu,n}_{M,s} & = & \bigwedge_{\xi<\nu} \psi^{\xi,n}_{M,s} \end{array}$$

where ψ_i lists all the literals in the variables $x_0, \ldots, x_{n-1}$ satisfied by s in M.
The sentence ϕ we need is $\psi^{\beta,0}_{M,\emptyset}$. Its quantifier-rank is clearly β.

- Clearly Eloise has a trivial strategy $J_E(\tau)$ in $G(M, \phi)$ (independently of τ), and this strategy is always a winning strategy.
- We now describe the strategy $J_A(\tau)$ of Abelard in $G(N, \phi)$.
- We call a position of the EF-game a **τ-position** if it arises while Abelard is playing τ.
- Suppose s is an assignment into M and s' an assignment into N, both with domain $\{x_0, \ldots, x_{n-1}\}$. We use $\boldsymbol{s \cdot s'}$ to denote the set of pairs $(s(x_i), s'(x_i))$, $i = 0, \ldots, n-1$. The **strategy** of Abelard is to play $G(N, \phi)$ in such a way that if the position at any point is $(\psi^{i,m-i}_{M,s}, s')$, then $s \cdot s'$ is a τ-position.

1. Suppose the position in $G(N, \phi)$ is $(\psi^{i,m-i}_{M,s}, s')$, $i > 0$, and the next move for Abelard in $\mathrm{EFD}_\beta(M, N)$ according to τ is $a \in M$.
2. The strategy of Abelard is to choose the latter conjunct of $\psi^{i,m-i}_{M,s}$. Then Abelard chooses the element $a \in M$ in the big conjunction move.
3. Now it is the turn of Eloise in $G(N, \phi)$ to choose some $b \in N$ as the value of x_{m-i} and that will be the next move of Eloise in $\mathrm{EFD}_\beta(M, N)$. The next position in $G(N, \phi)$ is

$$(\psi^{i-1,m-i+1}_{M,s(a/x_{m-i})}, s'(b/x_{m-i})). \tag{5}$$

4. The position $s(a/x_{m-i}) \cdot s'(b/x_{m-i})$ is still a τ-position in $\mathrm{EFD}_\beta(M, N)$.

1. Suppose the position in $G(N,\phi)$ is $(\psi^{i,m-i}_{M,s}, s')$, $i > 0$, and the next move for Abelard in $\mathrm{EF}_m(M,N)$ according to τ is $b \in N$.
2. The strategy of Abelard is to choose the former conjunct where he plays b as x_{m-i}. Now it is the turn of Eloise to choose some $a \in M$ in $G(N,\phi)$. The new position $s(a/x_{m-i}) \cdot s'(b/x_{m-i})$ is still a τ-position in $\mathrm{EF}_m(M,N)$. The next position in $G(N,\phi)$ is (5).
3. Finally the position is $(\psi^{0,m}_{M,s}, s')$. Note that $s \cdot s'$ is still a τ-position in $\mathrm{EF}_m(M,N)$. The game $\mathrm{EF}_m(M,N)$ has now ended. Abelard now chooses the first (in some fixed enumeration) literal conjunct of the formula $\psi^{0,m}_{M,s}$ that is not satisfied by s' in N, if any exist, otherwise he simply chooses the first conjunct.

- Suppose now τ was a winning strategy of Abelard. Then at the end of the game $s \cdot s'$ is a winning position for Abelard and therefore he is indeed able to choose a conjunct of the formula $\psi^{0,m}_{M,s}$ that is not satisfied by s' in N. He has won $G(N, \phi)$. QED

- If τ is a winning strategy of Abelard even in the game $\mathrm{EFD}_\beta(M \restriction L', N \restriction L')$ for some $L' \subset L$, then the separating sentence ϕ can be chosen so that its non-logical symbols are all in L'.
- If τ is such that Abelard plays only in M, we can make ϕ existential.
- If τ is such that Abelard plays only in N, we can make ϕ universal.
- If Abelard wins with τ even the harder game in which he has to win by finding an atomic (rather than literal) relation which holds in M but not in N, then we can take ϕ to be a positive sentence.

- Strategies in $\mathrm{EFD}_\beta(M, N)$ also reflect structural properties of M and N.
- If we know a strategy of Eloise in $\mathrm{EFD}_\beta(M_i, N_i)$ for $i \in I$, we can construct strategies of Eloise for EF games between products and sums of the models M_i and the respective products and sums of the models N_i. This can be extended to so-called κ-local functors [Feferman, 1972]. The situation is similar with tree-decompositions, e.g. [Grohe, 2007].
- EF games are known for infinitary logics, generalized quantifiers, and higher order logics.
- In modal logic the corresponding game is called the bisimulation game.

Team semantics

- A team is a set of assignments or a class of structures with assignments.
- EF game for teams: the players move and manipulate teams.
- EF game (on teams) for propositional logic [Hella and Väänänen, 2015].
- EF-game (on teams) for $L_{\omega_1\omega}$ [Väänänen and Wang, 2013].
- Adler and Immerman, 2001, for CTL and for reachability logic.

From quantifier-rank to formula-length: Hella-V. 2010

- Let $\mathcal{A}$ and $\mathcal{B}$ be classes of structures $(\mathcal{M}, s)$, s an assignment, of the same relational vocabulary, with $\mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(\mathcal{B})^2$, and let w be a positive integer. called the *rank* of the game.
- The game $\mathrm{EF}_w(\mathcal{A}, \mathcal{B})$ has two players, Abelard and Eloise.
- In the beginning the position is $(w, \mathcal{A}, \mathcal{B})$.
- Suppose the position after m moves is $(w_m, \mathcal{A}_m, \mathcal{B}_m)$, where $\mathrm{Dom}(\mathcal{A}_m) = \mathrm{Dom}(\mathcal{B}_m)$. Next:

[2] $\mathrm{Dom}(\mathcal{A})$ is the common domain of the assignments, e.g. $\{x_1, \ldots, x_n\}$.

Left splitting move: Abelard first chooses numbers u and v such that $1 \leq u, v < w$ and $u + v = w_m$. Then Abelard represents $\mathcal{A}_m$ as a union $\mathcal{C} \cup \mathcal{D}$. Now the game continues from the position $(u, \mathcal{C}, \mathcal{B}_m)$ or from the position $(v, \mathcal{D}, \mathcal{B}_m)$, and Eloise can choose which.

Right splitting move: Abelard first chooses numbers u and v such that $1 \leq u, v < w$ and $u + v = w_m$. Then Abelard represents $\mathcal{B}_m$ as a union $\mathcal{C} \cup \mathcal{D}$. Now the game continues from the position $(u, \mathcal{A}_m, \mathcal{C})$ or from the position $(v, \mathcal{A}_m, \mathcal{D})$, and Eloise can choose which.

Left supplementing move: Abelard chooses a natural number j and a choice function F for $\mathcal{A}_m$. Then the game continues from the position $(w_m - 1, \mathcal{A}_m(F/j), \mathcal{B}_m(\star/j))$.

Right supplementing move: Abelard chooses a natural number j and a choice function F for $\mathcal{B}_m$. Then the game continues from the position $(w_m - 1, \mathcal{A}_m(\star/j), \mathcal{B}_m(F/j))$.

The game ends in a position $(w_m, \mathcal{A}_m, \mathcal{B}_m)$ and Abelard wins if there is an atomic or a negated atomic formula ϕ that separates $\mathcal{A}$ and $\mathcal{B}$.

Theorem

Suppose $(\mathcal{A}, \mathcal{B})$ is a pair of classes of models of the same vocabulary. Then the following conditions are equivalent:

(1) *Abelard has a winning strategy in the game $EF_n(\mathcal{A}, \mathcal{B})$.*

(2) *There is a predicate logic sentence ϕ of size $\leq n$ which is true in every model in $\mathcal{A}$ and in no model in $\mathcal{B}$ which we denote*

$$(\mathcal{A}, \mathcal{B}) \models \phi.$$

Properties of $(\mathcal{A}, \mathcal{B}) \models \phi$

$(\mathcal{A}, \mathcal{B}) \models \neg\phi$	iff	$(\mathcal{B}, \mathcal{A}) \models \phi$
$(\mathcal{A}, \mathcal{B}) \models \phi \vee \psi$	iff	there are $\mathcal{A}_0$ and $\mathcal{A}_1$ such that
		$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, $(\mathcal{A}_0, \mathcal{B}) \models \phi$, $(\mathcal{A}_1, \mathcal{B}) \models \psi$
$(\mathcal{A}, \mathcal{B}) \models \phi \wedge \psi$	iff	there are $\mathcal{B}_0$ and $\mathcal{B}_1$ such that
		$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, $(\mathcal{A}, \mathcal{B}_0) \models \phi$, $(\mathcal{A}, \mathcal{B}_1) \models \psi$
$(\mathcal{A}, \mathcal{B}) \models \exists x_n \phi(x_n)$	iff	there is F with domain $\mathcal{A}$ such that
		$(\mathcal{A}(F/n), \mathcal{B}(\star/n)) \models \phi$
$(\mathcal{A}, \mathcal{B}) \models \forall x_n \phi(x_n)$	iff	there is F with domain $\mathcal{B}$ such that
		$(\mathcal{A}(\star/n), \mathcal{B}(F/n)) \models \phi$

Some simple applications of the team game

1. If ϕ is a propositional formula expressing the parity of strings $s \in \{0,1\}^n$, then the size of ϕ is at least n^2. (Wegener 1987)
2. If ϕ is an existential first order sentence expressing the property that all Boolean combinations of n unary predicates are non-empty, then the size of ϕ is at least $(n+1)2^n$.
3. If ϕ is an existential first order sentence expressing the property that the length of a linear order is at least n, then the size of ϕ is at least $2n-1$.
4. There is no formula shorter than 2^{n-1} that defines "In X, the truth values of $p_{k_0}, \ldots, p_{k_{n-2}}$ completely determine the truth value of $p_{k_{n-1}}$". Therefore it makes sense to adopt the dependence atom $=\!(p_{k_0}, \ldots, p_{k_{n-2}}, p_{k_{n-1}})$.

These are optimal values.

Summary

- The Evaluation Game, the Model Existence Game and the EF game go so deep into the essential concepts of logic such as truth, consistency, and separating models by sentences, that a lot of research in logic can be represented in terms of these games. (But this alone does not bring anything new.)
- The translations of the strategies between the games suggest a coherent uniform approach to syntax and semantics and at the same time a uniform approach to model theory and proof theory.
- The Evaluation Game and the EF game are oblivious to whether the models are finite or infinite, which gives them a useful role in computer science logic.
- Despite the vast literature on each of the three games separately, there seems to be a lot of potential for the study of their interaction as a manifestation of the Strategic Balance of Logic.

Thank you!

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