"It's all Greek to me"– on the pre-history of categorical logic

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This talk details a somewhat whimsical quest to give algebraic/categorical models of ancient Greek logic : precisely, the Stoic treatment of connectives expounded by Chrysippus, as understood (possibly misunderstood) by Hipparchus.

Our starting point is a well-documented disagreement between followers of Hipparchus, and those of Chrysippus, described in Plutarch's *Quaestiones Conviviales* and other sources. This was on the oddly specific question of how many distinct 'compound propositions' may be formed by conjunctions (or simply 'combinations') of ten atomic propositions. Chrysippus claimed to give an order of magnitude figure 'exceeding one million' (i.e. tens of myriads). In contrast, Plutarch reports — seemingly as a well-established fact — that Hipparchus and 'all the arithmeticians' had refuted Chrysippus, and established the precise number to be 103049.

The significance of this number remained unknown until 1994, when it was identified by Daniel Hough as the 10^{th} little Schröder number, and hence the number of rooted planar trees with ten leaves. Taking the very natural step of interpreting each tree leaf as an atomic proposition, and each branching as a connective, we arrive at the 'obvious interpretation' of such trees as compound propositions – thus giving a solution to a very long-standing unresolved question.

The combinatorics of Hipparchus' calculations is now well-understood, and several convincing attempts have been made to reconstruct how such a figure could be computed with ancient Greek mathematics. The logical interpretation is relatively less well-studied; a significant exception is the work of Suzanne Bobzien, who makes the case that his calculations were based on underlying conceptual errors; as part of this, propositions that should have been identified were instead counted separately. In particular, propositions arising via different (partial or total) bracketings of the same elementary propositions were incorrectly considered as entirely distinct.

This is of course familiar from modern categorical logic; we are often forced to use 'identical up to unique natural isomorphism' instead of some strict notion of equality.

A fun question (albeit with no historical justification whatsoever) is to consider what kinds of mathematical models would be required if the propositions that Bobzien claims as incorrectly counted separately were indeed related in this way? We give a system that does precisely this. We extend logical models that reflect the known sub-structural properties of Stoic logic[1] to include the assumptions apparently made by Hipparchus.

The result is 'freely generated' in the operadic sense, & may be used to label arbitrary facets of Stasheff's associahedra with functors corresponding to partial or total bracketings. We also exhibit natural transformations between them that live in a posetal category, and are therefore unique – thus accounting for the difference between 'equal' and 'equal up to unique natural isomorphism'.

This gives a range of diagrams, based on associahedra, guaranteed to commute (including, of course, MacLane's pentagon as a very special case). The components of these natural transformations are the 'congruential functions' familiar from proofs of undecidability in elementary arithmetic (from John Conway, Sergei Maslov, and other authors). We briefly mention some work of John Conway that gives a decidedly unexpected link with ancient Greek mathematics.

[1]Precisely, it is believed that the *Weakening* rule was not accepted; some sources also indicate that caution is needed concerning the *Contraction* rule.