Contextuality in logical form

Samson Abramsky

s.abramsky@ucl.ac.uk

UCL

Rui Soares Barbosa

rui.soaresbarbosa@inl.int

INL INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

Workshop on Springer Volume, UCL September 18–20

Why are we here?

Why are we here?

- The workshop and the volume are manifestations of a community.

Why are we here?

- The workshop and the volume are manifestations of a community.
- This is not an obvious community organised by topic.

Why are we here?

- The workshop and the volume are manifestations of a community.
- This is not an obvious community organised by topic.
- Topics represented include: logic, category theory, semantics of programming languages, game comonads and finite model theory, foundations of quantum mechanics and quantum computation, computational linguistics, . . .

Why are we here?

- The workshop and the volume are manifestations of a community.
- This is not an obvious community organised by topic.
- Topics represented include: logic, category theory, semantics of programming languages, game comonads and finite model theory, foundations of quantum mechanics and quantum computation, computational linguistics, ...
- I deeply believe that this is not haphazard, that there is a “community of spirit” in these endeavours, and also guiding ideas (Cf. Yoshiro Maruyama’s contribution to the volume).

Why are we here?

- The workshop and the volume are manifestations of a community.
- This is not an obvious community organised by topic.
- Topics represented include: logic, category theory, semantics of programming languages, game comonads and finite model theory, foundations of quantum mechanics and quantum computation, computational linguistics, ...
- I deeply believe that this is not haphazard, that there is a "community of spirit" in these endeavours, and also guiding ideas (Cf. Yoshiro Maruyama's contribution to the volume).
- My own recent work has led to some striking and unexpected connections between many of the strands represented here:
  - work with Rui and Amy on combining contextuality and causality (game semantics and contextuality)
  - work with Adam Ó Conghaile, Anuj and Rui, on connections between cohomological characterizations of contextuality, and constraint satisfaction and Weisfeiler-Leman.
  - work with Rui on a quantum duality, to be described here
  - more speculatively, ongoing work with Luca on arboreal categories, which I believe will make connections with game semantics and differential types

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

This focusses attention on the **non-classical aspects** of quantum theory.

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

This focusses attention on the **non-classical aspects** of quantum theory.

In particular, it brings **contextuality** into the picture.

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

This focusses attention on the **non-classical aspects** of quantum theory.

In particular, it brings **contextuality** into the picture.

- Contextuality is a key signature of non-classicality on quantum mechanics

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

This focusses attention on the **non-classical aspects** of quantum theory.

In particular, it brings **contextuality** into the picture.

- Contextuality is a key signature of non-classicality on quantum mechanics
- Non-locality (as in Bell's theorem) is a special case

Contextuality

The key foundational question in quantum computation is to characterize those information tasks where there is provable **quantum advantage** - i.e. the task can be performed better using quantum resources than with purely classical resources.

This focusses attention on the **non-classical aspects** of quantum theory.

In particular, it brings **contextuality** into the picture.

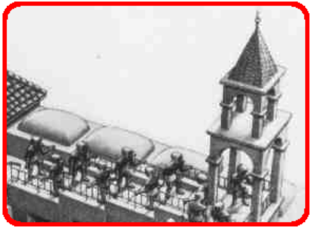
- Contextuality is a key signature of non-classicality on quantum mechanics
- Non-locality (as in Bell's theorem) is a special case
- Key role in many of the known cases of quantum advantage:
  shallow circuits, measurement-based quantum computation, VQE . . .

The essence of contextuality

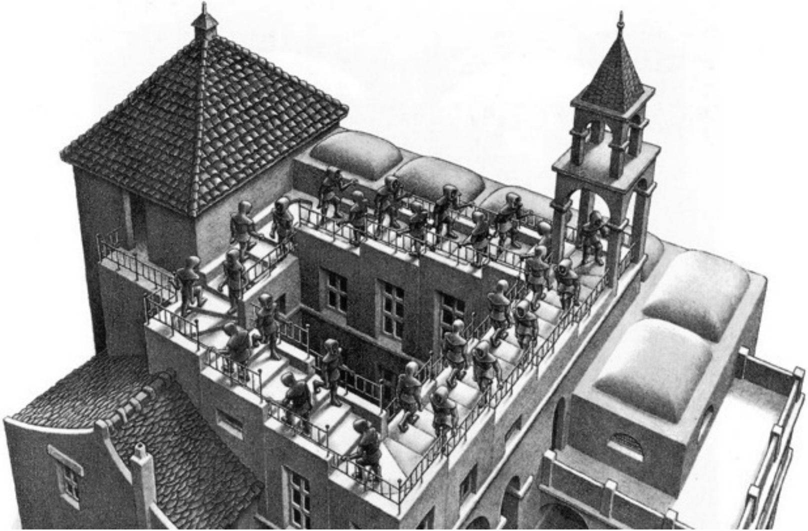
- Not all properties may be observed simultaneously.
- Sets of jointly observable properties provide **partial, classical snapshots**.
- Contextuality arises where there is a family of data which is

locally consistent but globally inconsistent

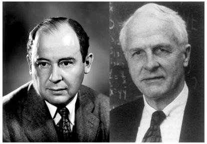
Contextuality Analogy: Local Consistency



Contextuality Analogy: Global Inconsistency

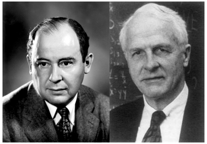


Background: traditional quantum logic



John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

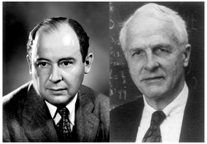
Background: traditional quantum logic



John von Neumann, in his seminal
Mathematical Foundations of Quantum Mechanics (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Background: traditional quantum logic

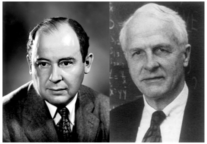


John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

Background: traditional quantum logic



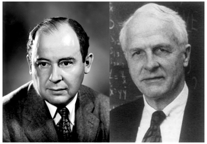
John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.

Background: traditional quantum logic



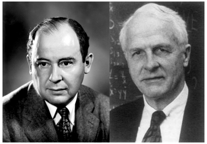
John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.

Background: traditional quantum logic



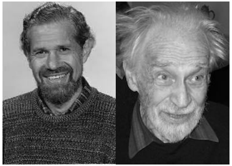
John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (1932), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

Projectors correspond 1–1 to the **closed subspaces** of Hilbert space.

Subsequently, Birkhoff and von Neumann, in *The Logic of Quantum Mechanics* (1936), proposed the lattice of closed subspaces as a non-classical logic to serve as the logical foundations of quantum mechanics.

- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Only commuting measurements can be performed together.
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?

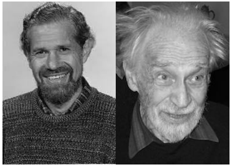
Quantum physics and logic



An alternative approach

Kochen & Specker (1965), ‘*The problem of hidden variables in quantum mechanics*’.

Quantum physics and logic

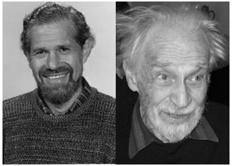


An alternative approach

Kochen & Specker (1965), '*The problem of hidden variables in quantum mechanics*'.

- The seminal work on contextuality used **partial Boolean algebras**.
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.

Quantum physics and logic



An alternative approach

Kochen & Specker (1965), '*The problem of hidden variables in quantum mechanics*'.

- The seminal work on contextuality used **partial Boolean algebras**.
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.

Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
- (partial) binary operations $\vee, \wedge : \odot \longrightarrow A$

such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

The key example: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined. This gives the category **pBA**.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

There is **no** pBA homomorphism $\mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}$.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

There is **no** pBA homomorphism $\mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}$.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

There is **no** pBA homomorphism $\mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}$.

- No assignment of truth values to all propositions that respects the logical operations on jointly testable propositions.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathsf{P}(\mathcal{H})$ its pBA of projectors.

There is **no** pBA homomorphism $\mathbf{P}(\mathcal{H}) \longrightarrow \mathbf{2}$.

- No assignment of truth values to all propositions that respects the logical operations on jointly testable propositions.
- Spectrum of a pBA cannot have *points*…

Conditions of impossible experience

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to **2**)*
2. *For some **propositional contradiction** $\varphi(\vec{x})$ and assignment $\vec{x} \mapsto \vec{a}$,*

$$A \models \varphi(\vec{a})$$

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $P(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $P(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Our local observations – **real observations** of **real measurements** – cannot be pieced together globally by reference to a single underlying objective reality. The values that they reveal are inherently contextual.

Conditions of impossible experience

Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

$$A \models \varphi(\vec{a})$$

Thus the event algebra $\mathrm{P}(\mathcal{H})$ of quantum mechanics cannot be interpreted globally in a consistent fashion.

Our local observations – **real observations** of **real measurements** – cannot be pieced together globally by reference to a single underlying objective reality. The values that they reveal are inherently contextual.

How can the world be this way? Still an ongoing debate, an enduring mystery . . .

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathrm{P}(\mathbb{C}^4)$, there is a set of five projectors (local Paulis) which generate a ***uniformly dense (infinite) subalgebra****.*

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathrm{P}(\mathbb{C}^4)$*, there is a set of five projectors (local Paulis) which generate a* ***uniformly dense (infinite) subalgebra****.*

Some elaborate geometry and algebra is used to show this.

Mysteries of partiality

Partial Boolean algebras can behave very differently to the total case.

It is a standard fact that every finitely-generated boolean algebra is finite.

Conway and Kochen (2002) show the following:

Theorem

In $\mathrm{P}(\mathbb{C}^4)$, there is a set of five projectors (local Paulis) which generate a ***uniformly dense (infinite) subalgebra****.*

Some elaborate geometry and algebra is used to show this.

Is there a “logical” proof?

Tensor product and the emergence of non-classicality

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ *on the category* **pBA** *such that the functor* $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ *is* ***strong monoidal*** *with respect to this structure, i.e. such that* $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$*?*

Tensor product and the emergence of non-classicality

As already remarked, the K-S property arises in $\mathsf{P}(\mathcal{H})$ when $\dim \mathcal{H} \geq 3$.

Note that $\mathbf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** *such that the functor* $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ *is* ***strong monoidal*** *with respect to this structure, i.e. such that* $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$*?*

A positive answer to this question would offer a complete logical characterisation of the Hilbert space tensor product, and provide the remaining step towards giving compositional logical foundations for quantum theory in a form useful for quantum information and computation.

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA’s.

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA's.

At first sight, this looks hopeless:

- classical Stone duality for boolean algebras B builds the Stone space of B from the **points**, i.e. homomorphisms $B \longrightarrow \mathbf{2}$
- by Kochen-Specker, for interesting cases of pBA's, there are no points!

Duality for partial Boolean Algebras?

Our aim is to get a duality theory for pBA's.

At first sight, this looks hopeless:

- classical Stone duality for boolean algebras B builds the Stone space of B from the **points**, i.e. homomorphisms $B \longrightarrow \mathbf{2}$
- by Kochen-Specker, for interesting cases of pBA's, there are no points!

We will instead generalize the **Tarski duality** for complete atomic Boolean algebras (CABAs)

CABAs

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \ .$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

Atoms are “state descriptions” or “possible worlds”.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Set}^{\mathrm{op}} \qquad \mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$$

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\cong}} \mathbf{Set}^{\mathrm{op}}$$

$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

Tarski duality

$$\mathbf{CABA} \;\underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}}\; \mathbf{Set}^{\mathrm{op}} \qquad (\cong)$$

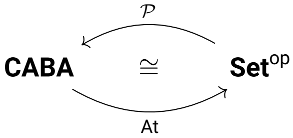
$\mathbf{At} : \mathbf{CABA}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

$$\mathrm{At}(h) : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$$

mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

Tarski duality



$\textbf{At} : \textbf{CABA}^{\textbf{op}} \longrightarrow \textbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

$$\text{At}(h) : \text{At}(B) \longrightarrow \text{At}(A)$$

mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

Duality for partial CABAs

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **partial CABA** is a complete, atomic partial Boolean algebra.

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

Definition (Atomic Boolean algebra)

A partial Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **partial CABA** is a complete, atomic partial Boolean algebra.

Note that $\mathrm{P}(\mathcal{H})$ is a partial CABA. Atoms are the rank-1 projectors (one-dimensional subspaces), i.e. the **pure states**.

Duality for partial CABAs: the idea

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the “non-commutative spaces” in this duality.

Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the “non-commutative spaces” in this duality.
- Morphism of graphs are certain relations, generalizing the functional relations which appear in classical Tarski duality.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y \# x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

Elements of X are called vertices, while unordered pairs $\{x, y\}$ with $x \# y$ are called edges.

Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y \# x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

A graph $(X, \#)$ has **finite clique cardinal** if all cliques are finite sets.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.
- If A is a Boolean algebra, then $\mathrm{At}(A)$ is the complete graph on the set of atoms ($\#$ is $\neq$).

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

- $\mathrm{At}(A)$ is the set of atomic events with an exclusivity relation.
- Can interpret these as *worlds of maximal information* and incompatibility between them.
- If A is a Boolean algebra, then $\mathrm{At}(A)$ is the complete graph on the set of atoms ($\#$ is $\neq$).

Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathrm{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

The key to reconstructing a partial CABA from its atoms lies in characterising such equalities,

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

The key to reconstructing a partial CABA from its atoms lies in characterising such equalities,

Proposition

Let K and L be cliques in $\mathrm{At}(A)$. Then $\bigvee K = \bigvee L$ iff $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \quad :\Leftrightarrow \quad K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;\; :\Leftrightarrow \;\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

Alternatively, take the double neighbourhood closures of cliques $K^{\#\#}$, yielding the sets U_a.

We can describe the algebraic structure of a partial CABA A from its graph of atoms:

- $0 = [\varnothing]$.
- $1 = [M]$ for any maximal clique M.
- $\neg[K] = [L]$ for any L maximal in $K^{\#}$, i.e. for any $L \# K$ such that $L \sqcup K$ is a maximal clique.
- $[K] \odot [L]$ iff there exist $K' \equiv K$ and $L' \equiv L$ such that $K' \cup L'$ is a clique.
- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

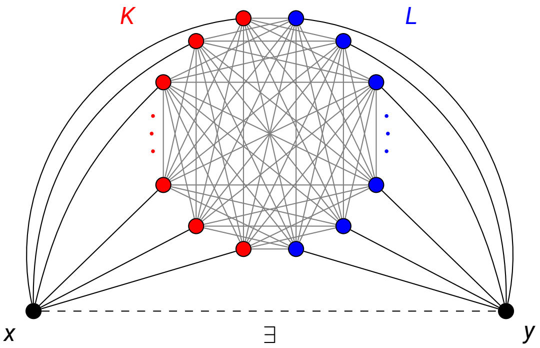
Which conditions on a graph $(X, \#)$ allow for such reconstruction?

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^\# \# L^\#$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^\# \subseteq y^\#$ implies $x = y$.



Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^\# \# L^\#$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^\# \subseteq y^\#$ implies $x = y$.

A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $y \# z$.

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^{\#} \subseteq y^{\#}$ implies $x = y$.

A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $y \# z$.
- Condition 1. is a weaker version of cotransitivity.
- Condition 2. eliminates redundant elements: cotransitive + 2. implies $\neq$.

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is a complete exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, □

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ *is a complete exclusivity graph.*

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, $\square$

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y$, $x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y$, $x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. xRy, $x'Ry'$, and $y \neq y'$ implies $x \neq x'$.

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y$, $x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. xRy, $x'Ry'$, and $y \neq y'$ implies $x \neq x'$.($x = x'$ implies $y = y'$. (functional))

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y, x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \, R \, y, x' \, R \, y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y$, $x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. xRy, $x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)
3. trivialises.

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x\ R\ y, x'\ R\ y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

For complete graphs:

1. $xRy, x'Ry'$, and $y \neq y'$ implies $x \neq x'$.
2. $R^{-1}(Y) = X$. (left-total)
3. trivialises.

Given $h : A \longrightarrow B$ define $y\ R\ x$ iff $y \leq h(x)$.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Proposition

Let X and Y be complete exclusivity graphs. Given $R : X \longrightarrow Y$ a morphism of complete exclusivity graphs, the function $h_R : \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$ given by $h_R([K]) := [L]$ where L is any clique maximal in $R^{-1}(K)$ is a well-defined partial CABA homomorphism.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad iff \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

Proposition

Let X and Y be complete exclusivity graphs. Given $R : X \longrightarrow Y$ a morphism of complete exclusivity graphs, the function $h_R : \mathcal{K}(Y) \longrightarrow \mathcal{K}(X)$ given by $h_R([K]) := [L]$ where L is any clique maximal in $R^{-1}(K)$ is a well-defined partial CABA homomorphism.

Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Global points

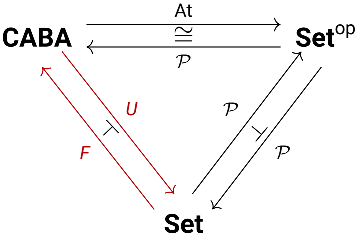
Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

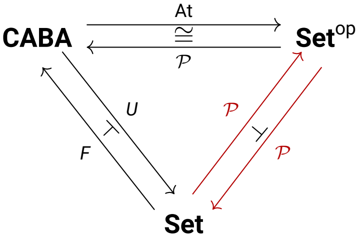
1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

The extensive literature on Kochen-Specker constructions is concerned with building graphs which have no such transversals, thus showing that the corresponding pBA's have no points.

Free-forgetful adjunction for CABAs

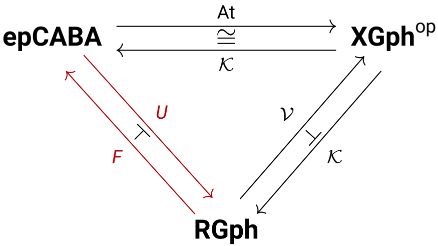


Free-forgetful adjunction for CABAs

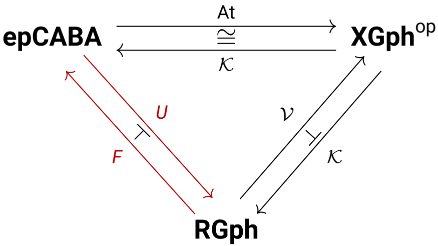


- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs

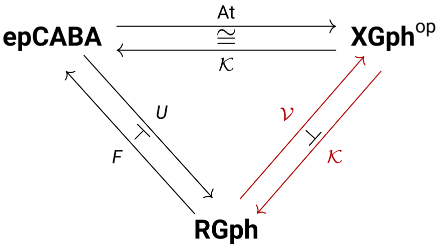


Free-forgetful adjunction for partial CABAs



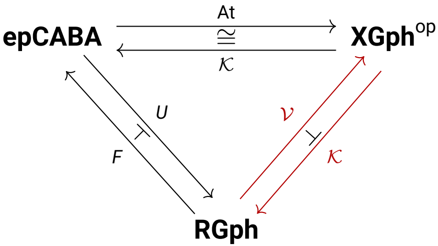
- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

Free-forgetful adjunction for partial CABAs



- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA.

Free-forgetful adjunction for partial CABAs



- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA. A compatibility $\langle P, \odot \rangle$ to a graph with vertices $\langle C, \gamma : C \longrightarrow \{0,1\} \rangle$ where C maximal compatible set, and edges

$$\langle C, \gamma \rangle \# \langle D, \delta \rangle \qquad \text{iff} \qquad \exists x \in C \cap D.\ \gamma(x) \neq \delta(x)\ .$$