Logic and structure at the borders of paradox

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

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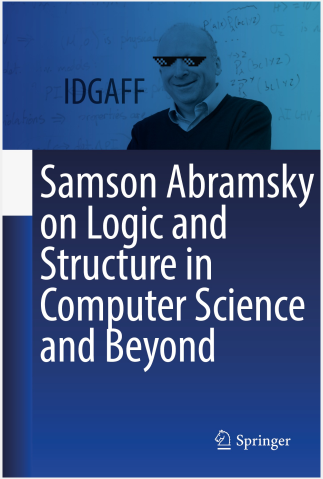
The volume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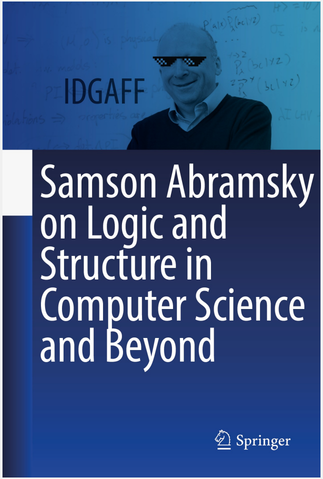

Quantum Physics

[Submitted on 22 Apr 2021]

Closing Bell: Boxing black box simulations in the resource theory of contextuality

Rui Soares Barbosa, Martti Karvonen, Shane Mansfield

This chapter contains an exposition of the sheaf-theoretic framework for contextuality emphasising resource-theoretic aspects, as well as some original results on this topic. In particular, we consider functions that transform empirical models on a scenario S to empirical models on another scenario T, and characterise those that are induced by classical procedures between S and T corresponding to 'free' operations in the (non-adaptive) resource theory of contextuality. We proceed by expressing such functions as empirical models themselves, on a new scenario built from S and T. Our characterisation then boils down to the non-contextuality of these models. We also show that this construction on scenarios provides a closed structure in the category of measurement scenarios.

Comments: 36 pages. To appear as part of a volume dedicated to Samson Abramsky in Springer's Outstanding Contributions to Logic series
Subjects: **Quantum Physics (quant-ph)**; Logic in Computer Science (cs.LO); Category Theory (math.CT)
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(or arXiv:2104.11241v1 **[quant-ph]** for this version)

Context of this talk

- Samson’s quantum turn (QCM in 2004),

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- and then contextuality (2011):

  '*The sheaf-theoretic structure of non-locality and contextality*'
  Abramsky & Brandenburger, NJP 2011.

  ⋮

  '*Contextuality: at the borders of paradox*'
  Abramsky, Categories for the working philosopher 2020.

This talk

Recent work with Samson on algebraic-logic view of contextuality,
revisiting Kochen & Specker's partial Boolean algebras.

'*The logic of contextuality*'
Abramsky & B, CSL 2021.

'*Contextuality in logical form: Duality for transitive partial CABAs*'
Abramsky & B, TACL 2022, QPL 2023.

Joint work in progress with Samson Abramsky, Martti Karvonen, Raman Choudhary, ...

Contextuality and advantage in quantum computation

- Central object of study of quantum information and computation theory:
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- A range of examples are known and have been studied . . . but a systematic understanding of the scope and structure of quantum advantage is lacking.
- A hypothesis: this is related to **non-classical** features of quantum mechancics.
- **Contextuality** is a quintessential marker of non-classicality, an empirical phenomenon distinguishing QM from classical physical theories.

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- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  - Contextuality analysis: Aasnæss, Forthcoming, 2020.

The essence of contextuality

- Not all properties may be observed simultaneously.
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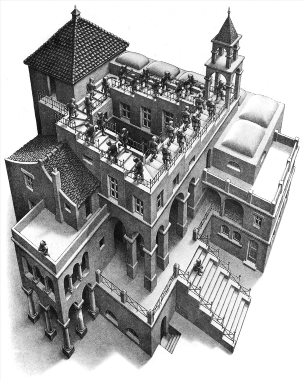

M. C. Escher, *Ascending and Descending*

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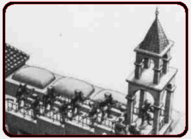

Local consistency

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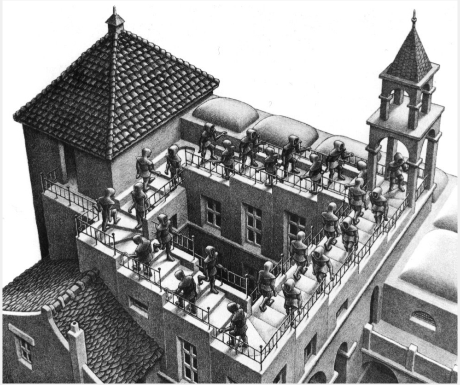

Local consistency *but* **Global inconsistency**

Logic and quantum theory

From states to properties

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of 'conserving the validity of all formal rules' [...]. Now we begin to believe that it is not the *vectors* which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the *linear closed subspaces* [von Neumann (1935) as quoted in Birkhoff (1966)]

From classical to quantum

John von Neumann (1932), ‘*Mathematische Grundlagen der Quantenmechanik*’.

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Classical mechanics

- Described by **commutative** C^*-algebras or von Neumann algebras.
- By Gel'fand duality, these are algebras of continuous (or measurable) functions on topological spaces, the state spaces.

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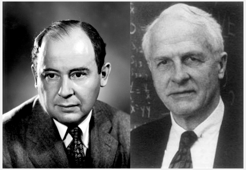
- Measurements are self-adjoint operators.
- Quantum properties or propositions are **projectors** (dichotomic measurements):

$$p : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.

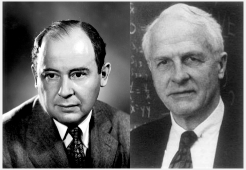
Traditional quantum logic

Birkhoff & von Neumann (1936), '*The logic of quantum mechanics*'.



- The lattice $P(\mathcal{H})$, of projectors on a Hilbert space $\mathcal{H}$, as a non-classical logic for QM.

Quantum physics and logic



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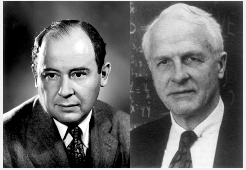

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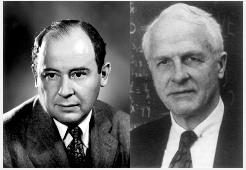

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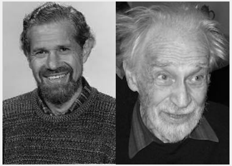
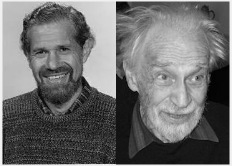
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Taking the *phenomenological* requirement seriously:
  in QM, only **commuting** measurements can be performed together.

  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?



An alternative approach

Kochen & Specker (1965), ‘*The problem of hidden variables in quantum mechanics*’.



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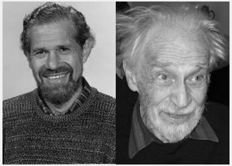
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.



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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Partial Boolean algebras

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- constants $0, 1 \in A$
- a unary operation $\neg : A \longrightarrow A$
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satisfying the usual axioms: $\langle A, \vee, 0\rangle$ and $\langle A, \wedge, 1\rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap\rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

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Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Partial Boolean algebras

A more concrete formulation of the defining axioms is:

- operations preserve commeasurability: for each n-ary operation f,

$$\frac{a_1 \odot c, \; \ldots, \; a_n \odot c}{f(a_1, \ldots, a_n) \odot c}$$

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- for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied, e.g.

$$\frac{a \odot b}{a \wedge b = b \wedge a} \qquad \frac{a \odot b, \; a \odot c, \; b \odot c}{a \wedge (b \vee c) \; = \; (a \wedge b) \vee (a \wedge c)}$$

The category **pBA**

Morphisms of partial Boolean operations are maps preserving commeasurability, and the operations wherever defined.This gives a category **pBA**.

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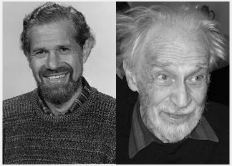
Abramsky & B (2020), ‘*The logic of contextality*’.

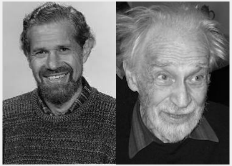
- We give a direct construction of colimits.
- More generally, we show how to freely generate from a given partial Boolean algebra A a new one satisfying prescribed additional commeasurability relations $\circ$, denoted $A[\circledcirc]$.



Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

Contextuality, or the Kochen–Specker theorem



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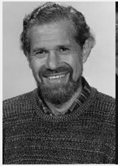
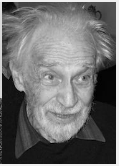

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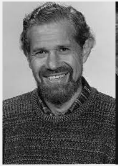
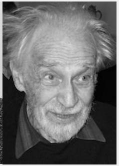

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- No assignment of truth values to all propositions which respects logical operations on jointly testable propositions.

An apparent contradiction

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But note that **BA** is an equational variety of algebras over **Set**.

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- The cone in **BA** from $\mathcal{C}(A)$ to B is also a cone in **pBA**, hence there is $A \longrightarrow B$!

But note that **BA** is an equational variety of algebras over **Set**.

As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. This Boolean algebra does **not** have a homomorphism to **2**.

An apparent contradiction

- **BA** is a full subcategory of **pBA**.
- Given a partial Boolean algebra A, consider the diagram $\mathcal{C}(A)$ of its Boolean subalgebras.
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If a partial Boolean algebra A has no homomorphism to **2**, then $\varinjlim_{\mathbf{BA}} \mathcal{C}(A) = \mathbf{1}$.

Kochen–Specker and conditions of ‘impossible’ experience

We could say that such a diagram is “implicitly contradictory”: in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

Contextuality: partial views are locally consistent but globally inconsistent!

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Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property, i.e. it has no morphism to* **2**.
2. *The colimit in* **BA** *of the diagram* $\mathcal{C}(A)$ *of boolean subalgebras of A is* **1**.

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4. *There is a Boolean term* $\varphi(\vec{x})$ *with* $\varphi(\vec{x}) \equiv_{\mathrm{Bool}} 0$ *and an assignment* $\vec{x} \mapsto \vec{a}$ *such that* $\varphi(\vec{a})$ *is well-defined and equals* 1.

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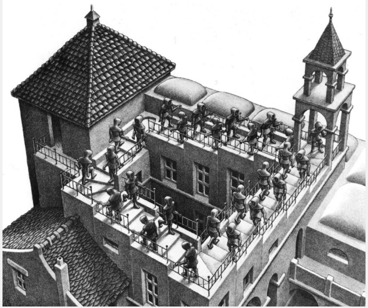

At the borders of paradox:
the contradiction is never directly observed!

Quantum realisation

$$((a \oplus d) \oplus (b \oplus c)) \oplus ((a \oplus b) \oplus (a \oplus d))$$

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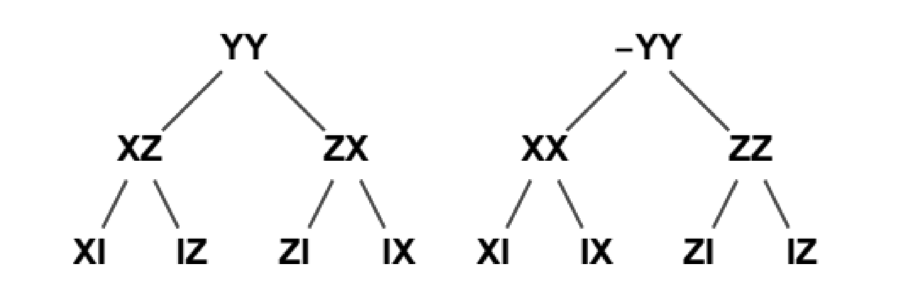
$$\langle \{0, 1\}, \oplus \rangle \quad \longleftrightarrow \quad \langle \{1, -1\}, \cdot \rangle$$



Compound systems

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DISCUSSION OF PROBABILITY RELATIONS BETWEEN SEPARATED SYSTEMS

By E. SCHRÖDINGER

[Communicated by Mr M. Born]

[*Received* 14 August, *read* 28 October 1935]

1. When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or ψ-functions) have become entangled. To disentangle them we must

Question

How do properties of systems compose?

A [first] tensor product by generators and relations

Heunen & van den Berg show that **pBA** has a monoidal structure:

$$A \otimes B := \text{colim}\,\{C + D \mid C \in \mathcal{C}(A), D \in \mathcal{C}(B)\}$$

where $C + D$ is the coproduct of Boolean algebras.

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We can use our construction to give an explicit generators-and-relations description.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

Tracking the quantum mechanical tensor product?

- There is an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the obvious embeddings

$$\mathsf{P}(\mathcal{H}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K}) :: p \longmapsto p \otimes 1$$
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  - The gap is that more relations hold in $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ than in $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K})$.
  - Nevertheless, this result is suggestive.
    It poses the challenge of finding a stronger notion of tensor product.

Mysteries of partiality

A slight detour: free partial Boolean algebra

Free partial Boolean algebra on a reflexive graph $(X, \frown)$
(a 'graphical' measurement scenario).

- Generators $G := \{\imath(x) \mid x \in X\}$.
- Pre-terms P: closure of G under Boolean operations and constants.

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- $F(X) = T/\equiv$, with obvious definitions for $\odot$ and operations.

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- So, for $X \subseteq A$, the map $F(X, \odot_X) \longrightarrow \langle X \rangle_A$ need not be surjective!
- How come? The reason is that **new compatibilities** arise!

$$\text{not just} \qquad \frac{t \odot u, \; t \odot v, \; u \odot v}{(t \wedge u) \odot v}$$

A more expressive tensor product

- Consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$.
- to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$.
- we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not

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  Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

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This amounts to composing with the reflection to **epBA**; $\boxtimes \ := \ X \ \circ \ \otimes$.
Explicitly, we define the logical exclusivity tensor product by

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- This is sound for the Hilbert space model.
- It remains to be seen how close it gets us to the full Hilbert space tensor product.

A limitative result

- Can extending commeasurability by a relation $\circledcirc$ induce the K-S property in $A[\circledcirc]$ when it did not hold in A?

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $\circledcirc \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[\circledcirc]$ is K-S.

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In particular, $\mathrm{P}(\mathbb{C}^2) \boxtimes \mathrm{P}(\mathbb{C}^2)$ does not have the K-S property.

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Let A be a partial Boolean algebra, and $\circledcirc \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[\circledcirc]$ is K-S.

Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would imply that the LE tensor product $A \boxtimes B$ never induces a K-S paradox if none was present in A or B.

In particular, $\mathrm{P}(\mathbb{C}^2) \boxtimes \mathrm{P}(\mathbb{C}^2)$ does not have the K-S property.

We need an even stronger tensor product to track the emergent complexity in the quantum case!

A simpler problem

Restrict the problem

Forget some structure:

- Parity or XOR/NOT logic
- i.e. $(\neg, \oplus)$-fragment
- this is the ‘linear (or actually *affine*) part’ of Boolean algebra

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Consider the Pauli operators

- $P \in (\mathbb{C}^2)^{\otimes n}$
- s.t. $P = \alpha(P_1 \otimes \cdots \otimes P_n)$,
  with $P_i \in \{X, Y, Z, \mathbf{1}\}$, $\alpha \in \{1, -1, i, -i\}$

Boolean affine space

Boolean affine space $\langle A, 0, \oplus, \neg \rangle$:

- a set A
- constant $0 \in A$
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Note that $\neg a = a \oplus 1$, so we could define this with 1.

Partial Boolean affine space

Partial Boolean affine space $\langle A, \odot, 0, \oplus, \neg \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constant $0 \in A$
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But also: (projectors associated with) n-Pauli operators, $\mathcal{P}_n \preceq \mathrm{P}((\mathbb{C}^2)^{\otimes n})$

Recovering the Paulis

$$\frac{t \odot u,\ t \odot v,\ u \odot v}{(t \oplus u) \odot v}$$

Crucially, Paulis either **commute** or **anticommute**

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This fully characterises commeasurability of '$\oplus$'s of Paulis, without needing to inspect the concrete Paulis. That is, whether $\phi(\vec{a})$ is commeasurable with b does not depend on the concrete **a** and b but only on the commeasurability structure of $\{a_1, \ldots, a_n, b\}$.

This addresses the compatibility issue in reconstructing $\mathcal{P}_n$ as a partial Boolean affine space.

Thank you!

Questions...

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