Towards a Classification of Contextuality

Hardy is almost everywhere
Non-Locality without inequalities
for almost all entangled multipartite states

Samson Abramsky, Carmen Maria Constantin, Shenggang Ying

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Towards a Classification of Contextuality

(Weak < Logical < Strong)

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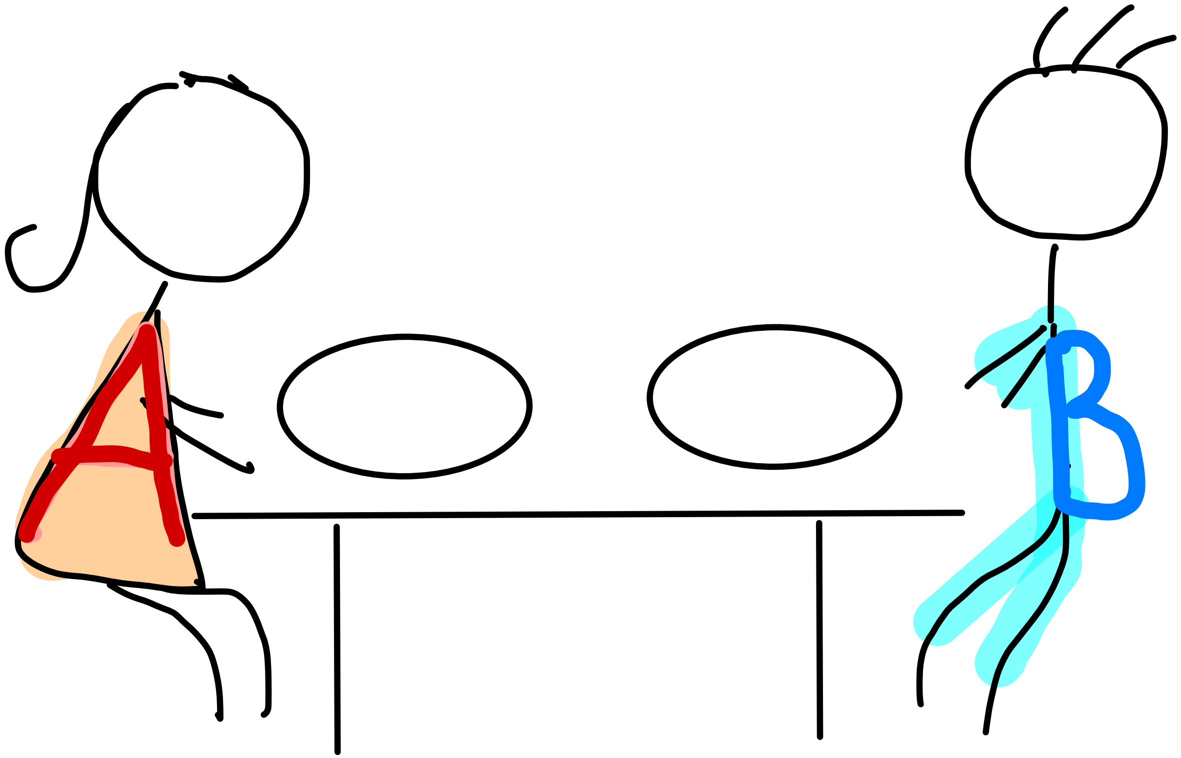
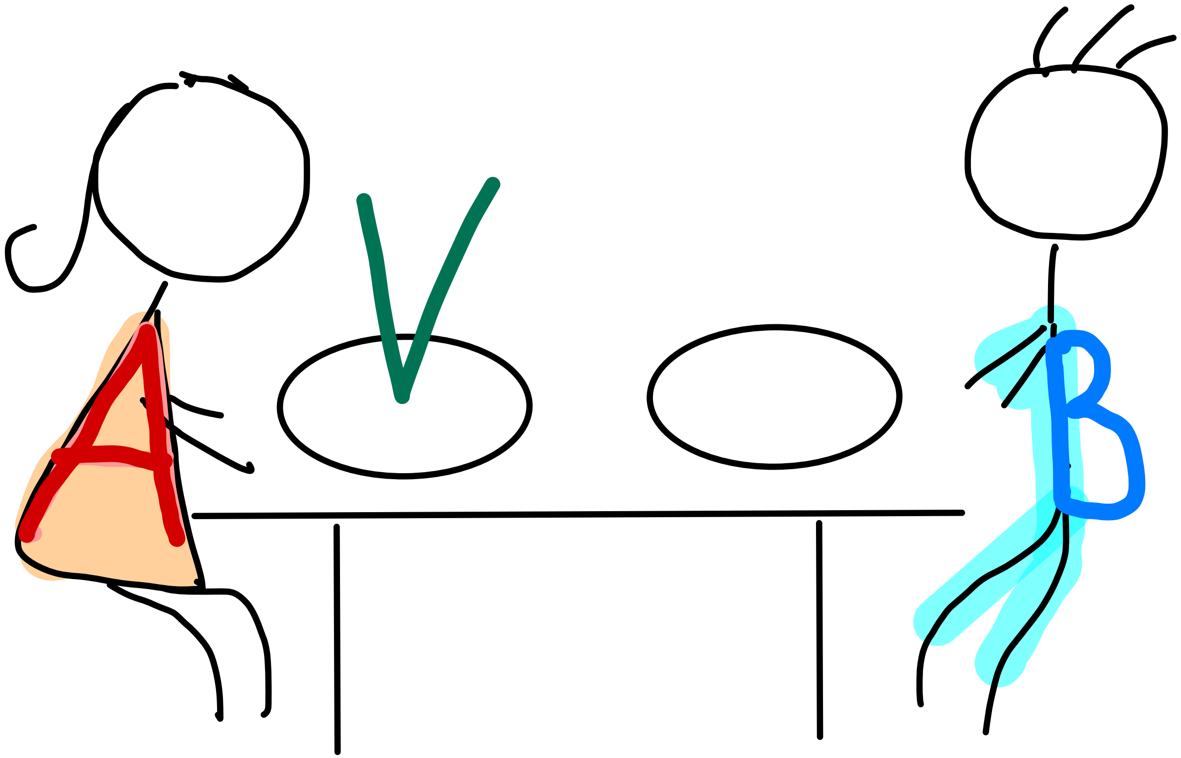
Setting the Scene



Setting the Scene



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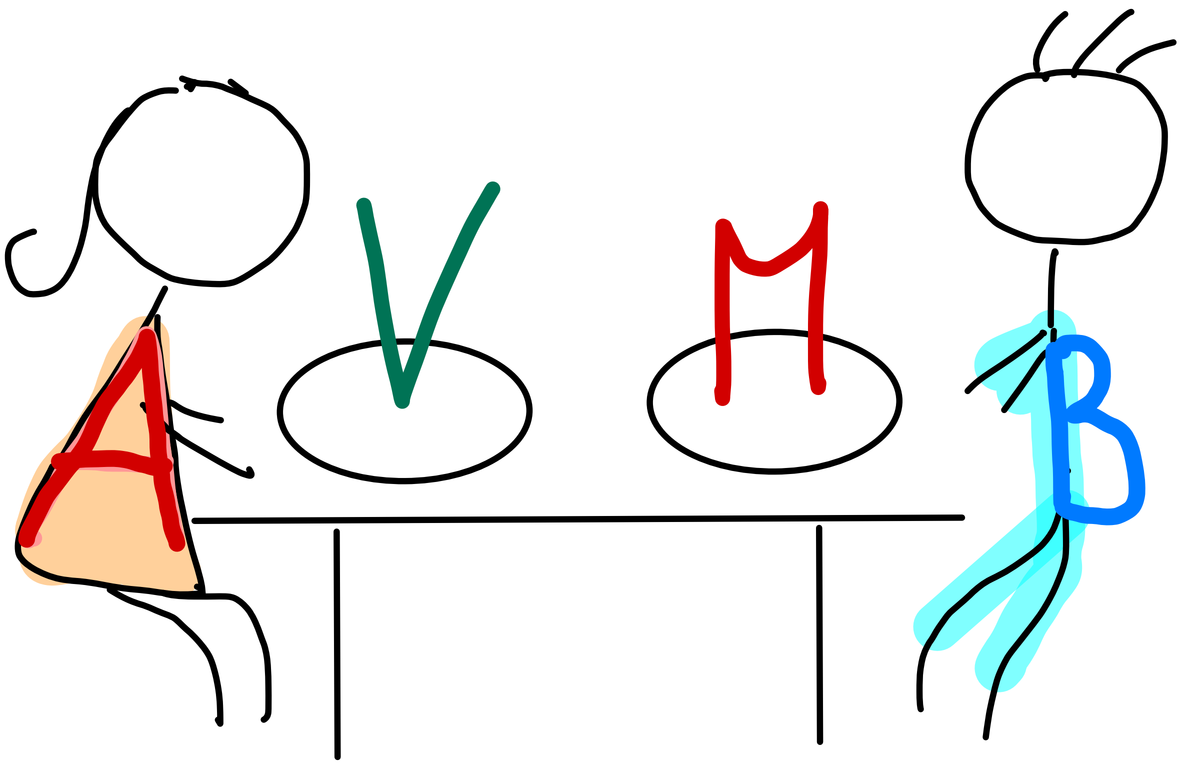

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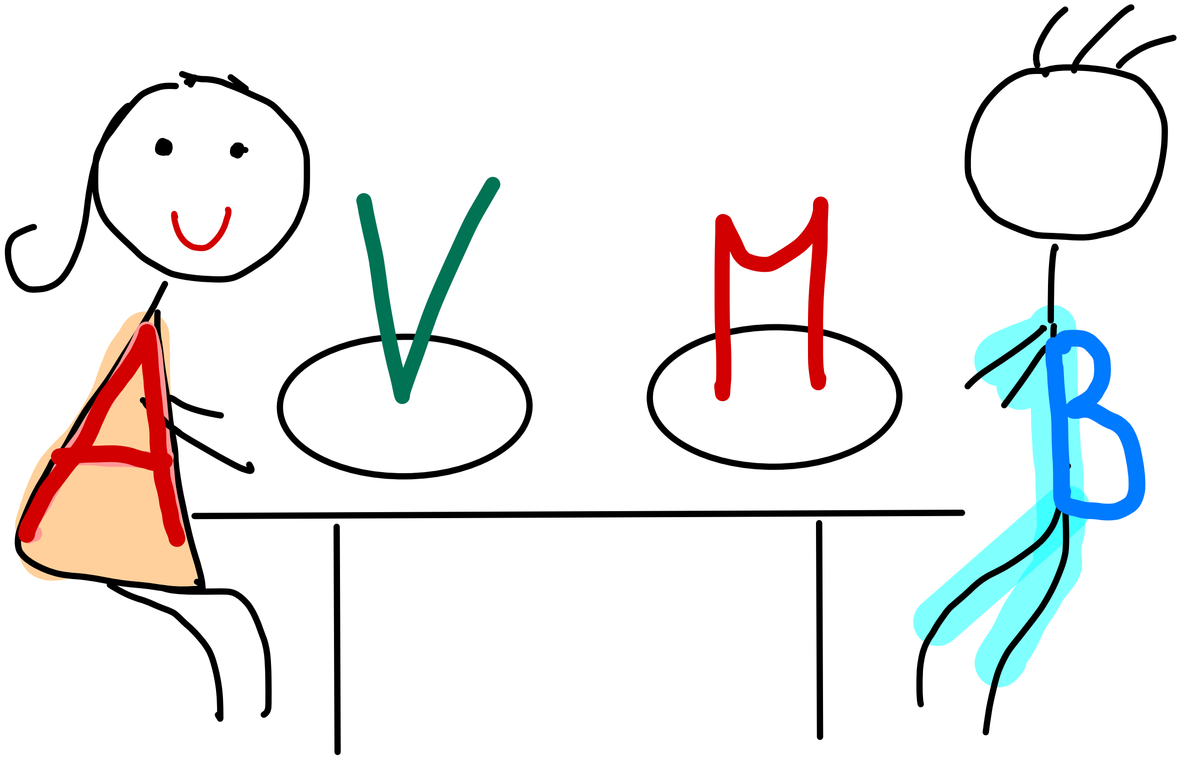
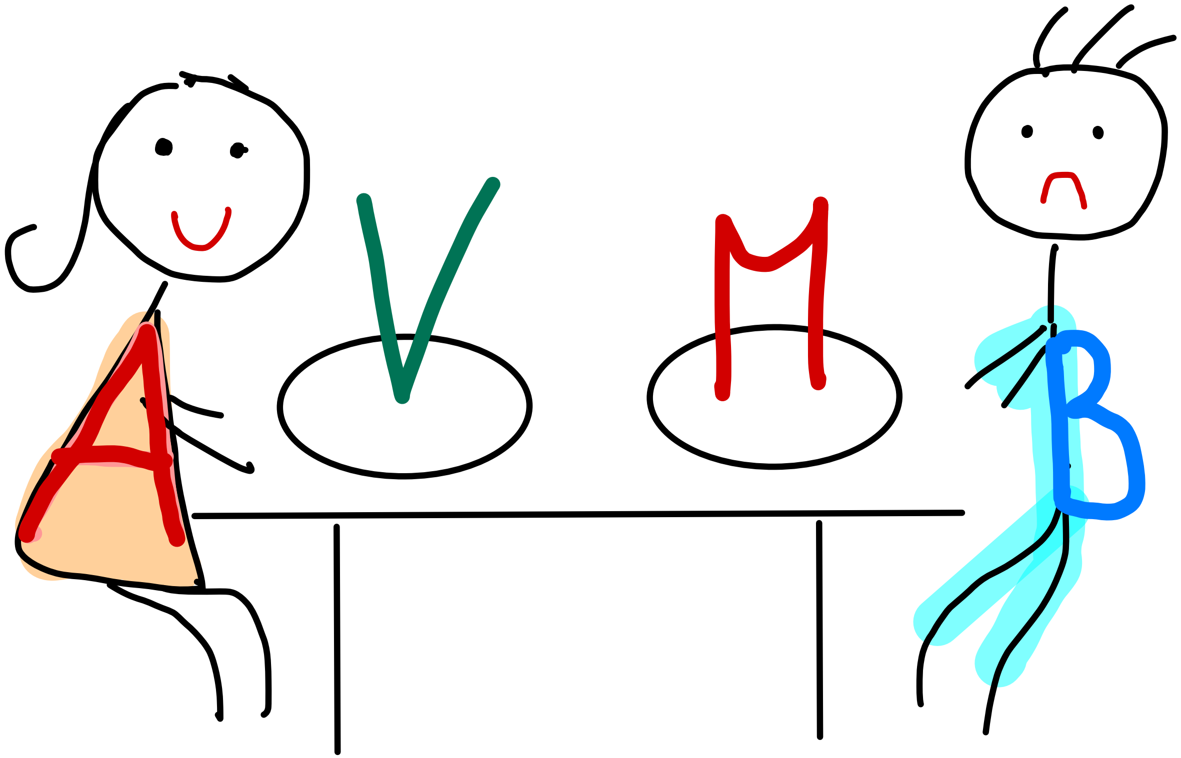
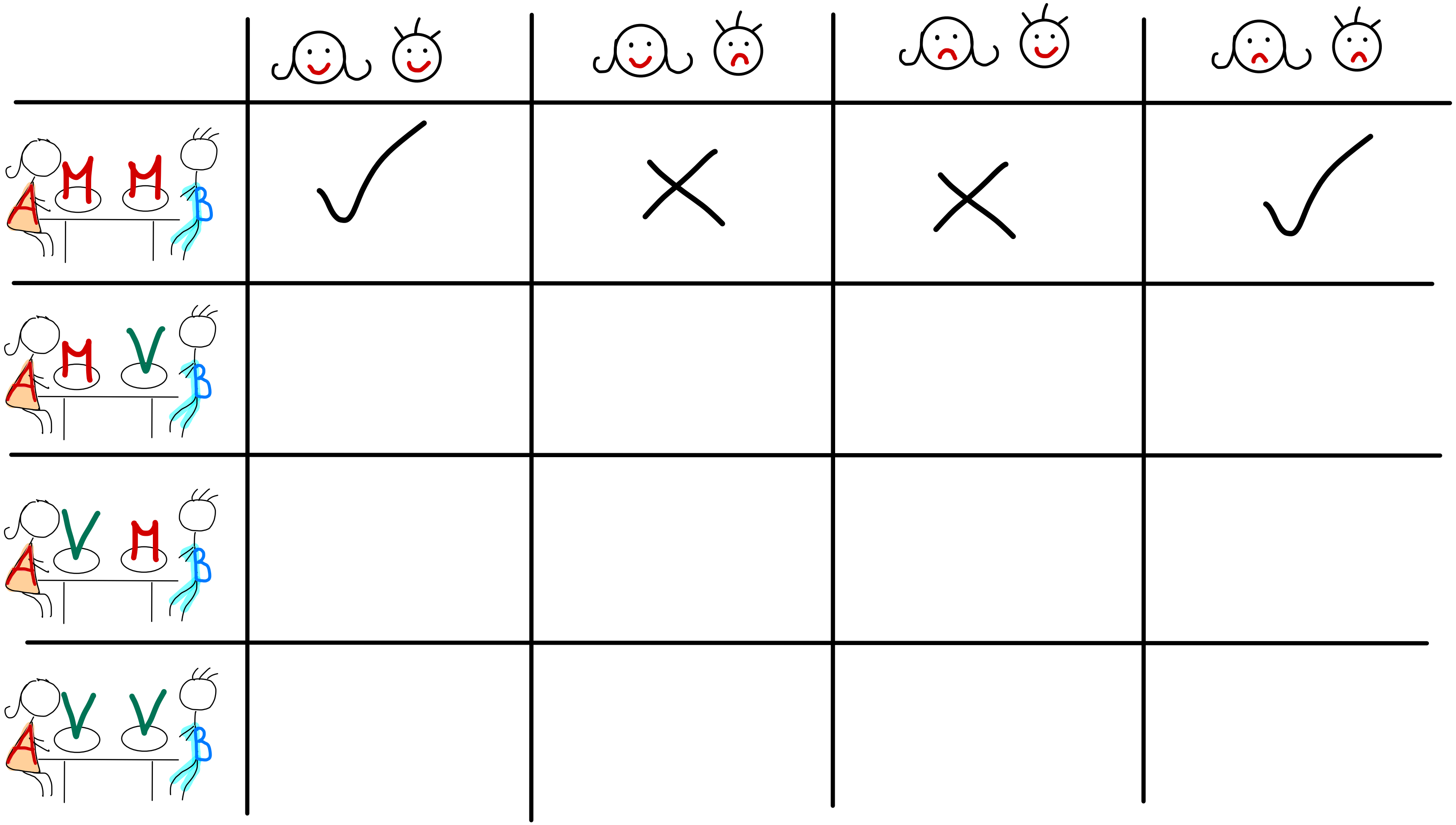


Setting the Scene



Recording Empirical Observations



Idea: Keep track of what CAN / CAN'T happen
(NOT how likely it is to happen)

Empirical Models

	+ +	+ −	− +	− −
M M				
M V				
V M				
V V				

Empirical Models

	+ +	+ −	− +	− −
M M				
M V				
V M				
V V				

contexts

Empirical Models

	+ +	+ −	− +	− −
M M				
M V				
V M				
V V				

contexts

local sections

Empirical Models

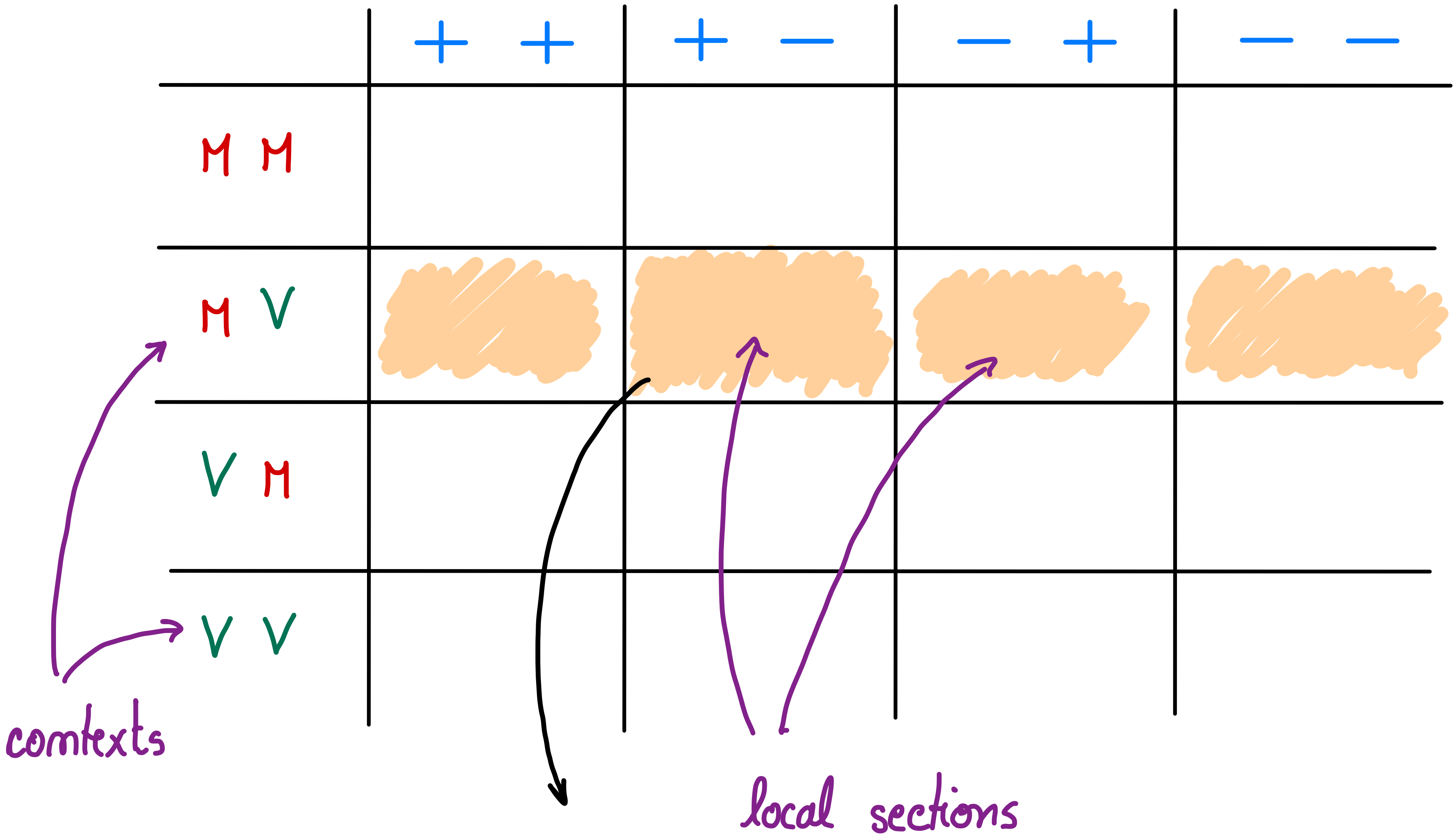
	+ +	+ −	− +	− −
M M				
M V				
V M				
V V				

contexts

local sections

This cell specifies a function $s : \{M_A, V_B\} \to \{+, -\}$

Empirical Models



This cell specifies a function $s : \{M_A, V_B\} \longrightarrow \{+, -\}$

AND an atomic logical formula $\varphi_s = M_A \wedge \neg V_B$

Empirical Models

	+ +	+ −	− +	− −
M M				
M V	1	1	1	0
V M				
V V				

contexts

local sections

empirical data

(determines the SUPPORT of the E.M.)

Global to Local

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M				
M V				
V M				
V V				

Global sections:

M_A	M_B	V_A	V_B
$+$	$+$	$-$	$+$

;

M_A	M_B	V_A	V_B
$-$	$-$	$-$	$-$

Global to Local

	+ +	+ −	− +	− −
M M	1	0	0	1
M V	1	0	0	1
V M	1	0	0	1
V V	1	0	0	1

Global sections:

M_A	M_B	V_A	V_B
+	+	−	+

;

M_A	M_B	V_A	V_B
−	−	−	−

Global to Local

	+ +	+ −	− +	− −
M M	1	0	0	1
M V	1	0	0	1
V M	1	0	0	1
V V	1	0	0	1

Global sections:

M_A	M_B	V_A	V_B
+	+	−	+

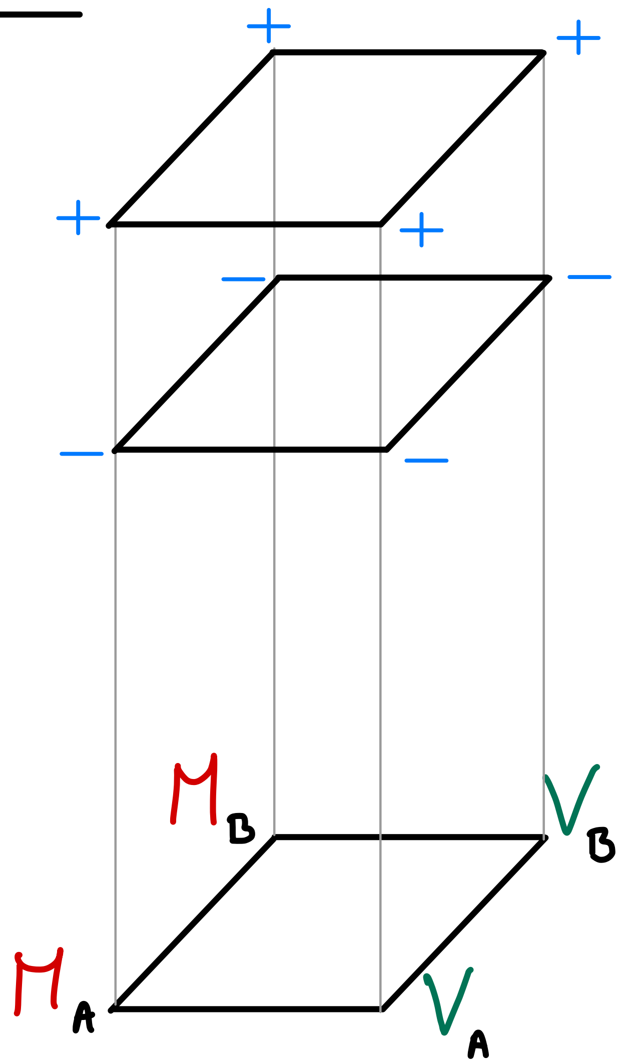
;

M_A	M_B	V_A	V_B
−	−	−	−

may be inconsistent with the support

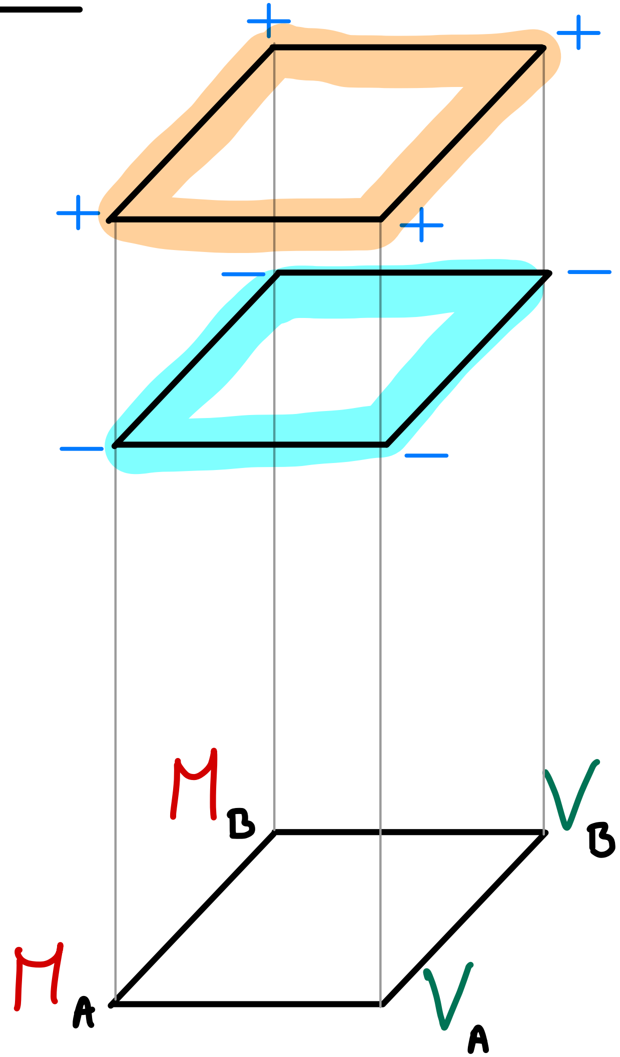
Local to Global

	+ +	+ −	− +	− −
M M	1			1
M V	1			1
V M	1			1
V V	1			1



Local to Global

	+ +	+ −	− +	− −
M M	1			1
M V	1			1
V M	1			1
V V	1			1



Global sections:

M_A	M_B	V_A	V_B
+	+	+	+

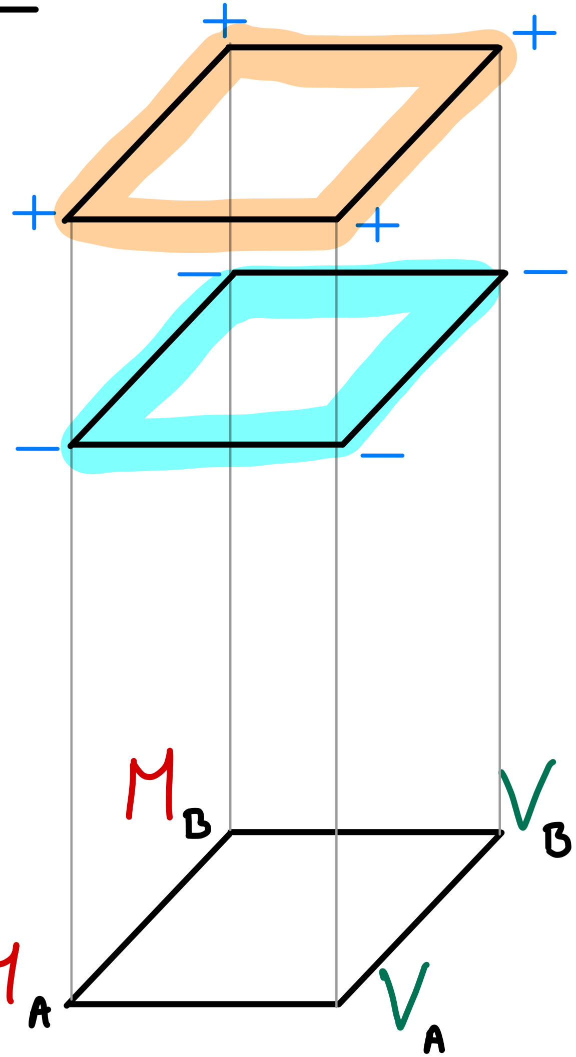
;

M_A	M_B	V_A	V_B
−	−	−	−

✓

Local to Global

	+ +	+ −	− +	− −
M M	1			1
M V	1			1
V M	1			1
V V	1			1



Global sections:

M_A	M_B	V_A	V_B
+	+	+	+

;

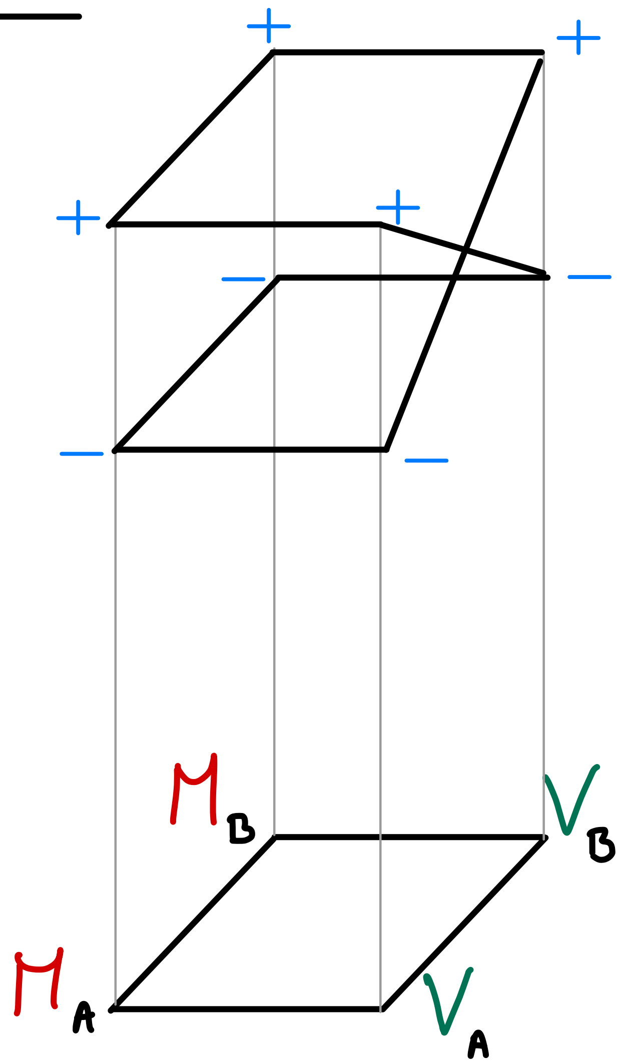
M_A	M_B	V_A	V_B
−	−	−	−

Not contextual

(in the logical sense)

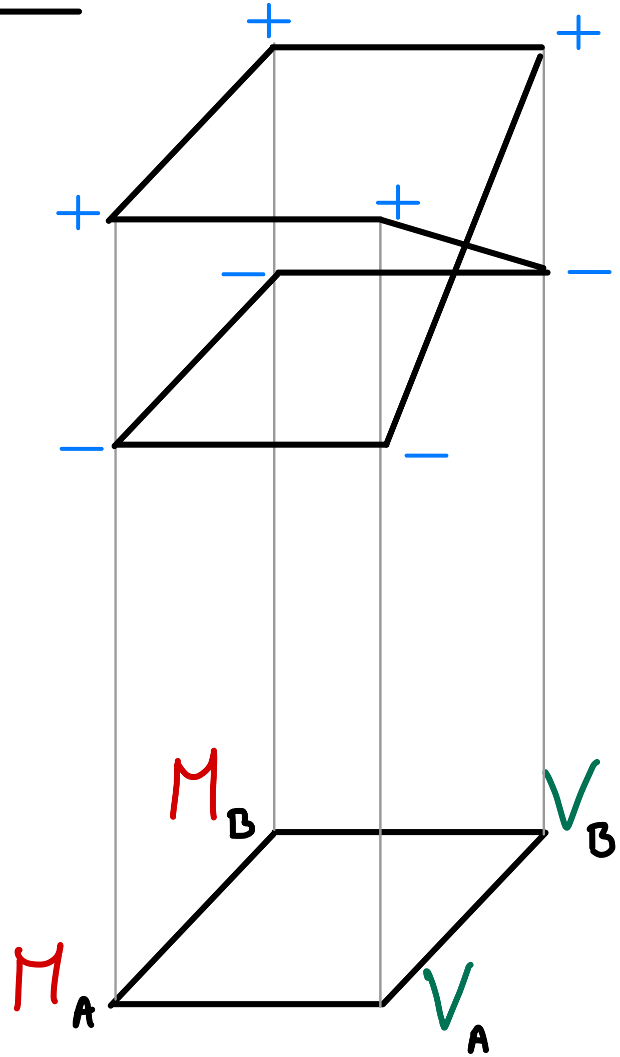
Local to Global

	+ +	+ −	− +	− −
M M	1			1
M V	1			1
V M	1			1
V V		1	1	



Local to Global

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M	1			1
M V	1			1
V M	1			1
V V		1	1	

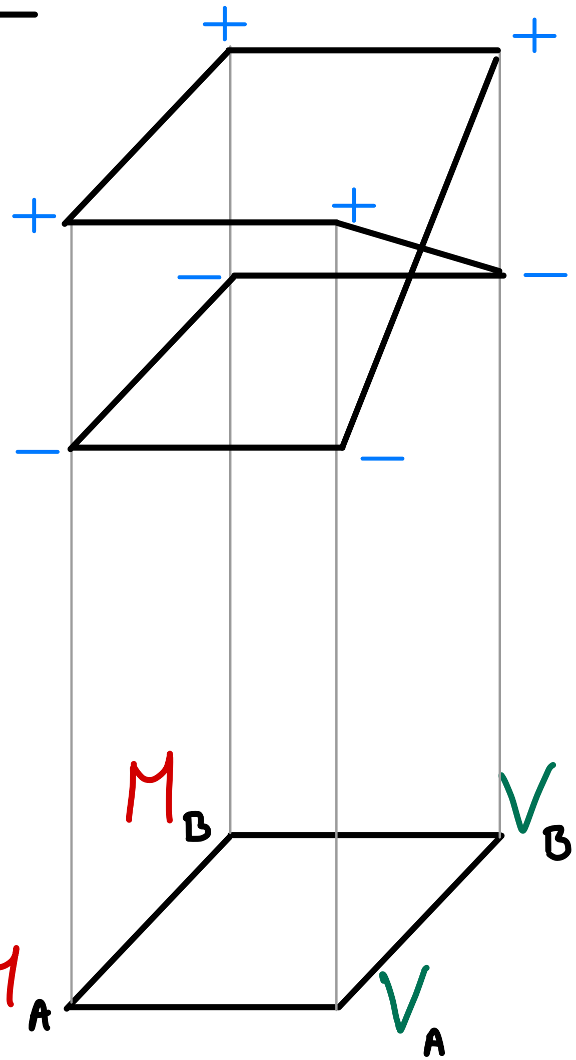


Global sections:

M_A	M_B	V_A	V_B
×	×	×	×

Local to Global

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M	1			1
M V	1			1
V M	1			1
V V		1	1	



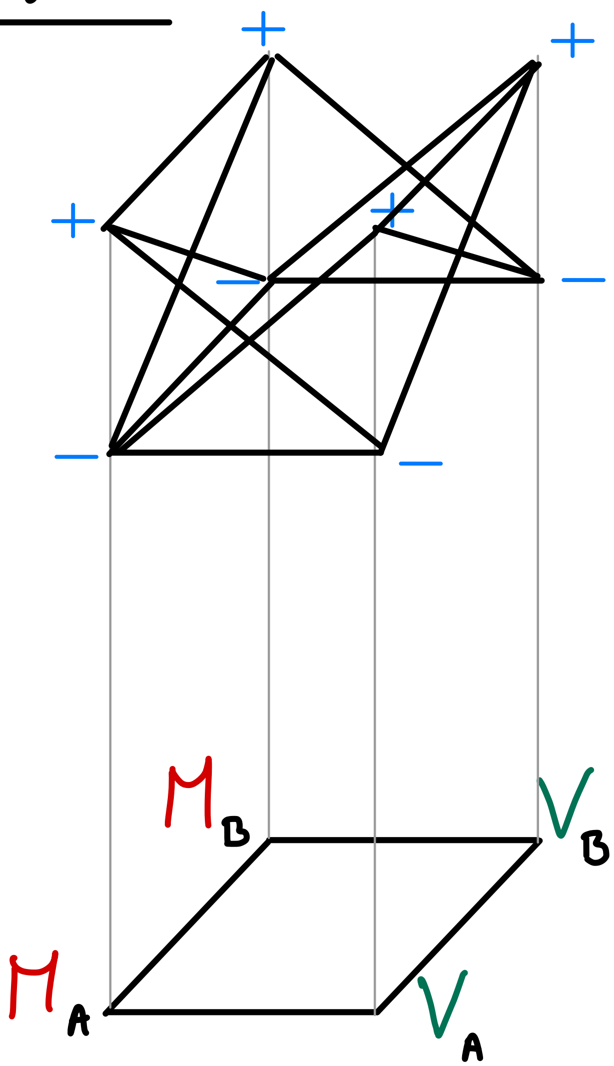
Global sections:

M_A	M_B	V_A	V_B
×	×	×	×

" Strong Contextuality "

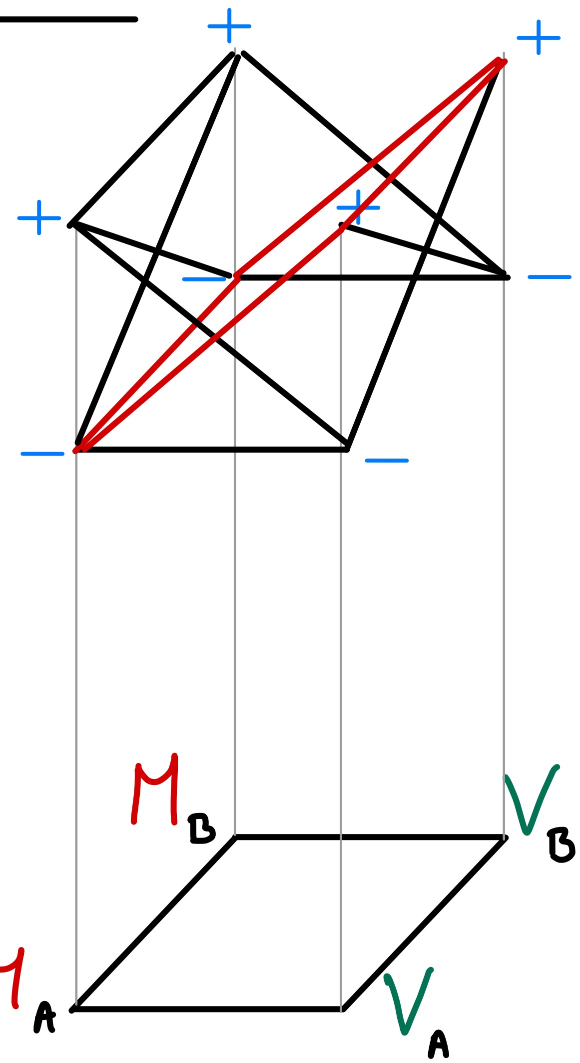
Local to Global

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	



Local to Global

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

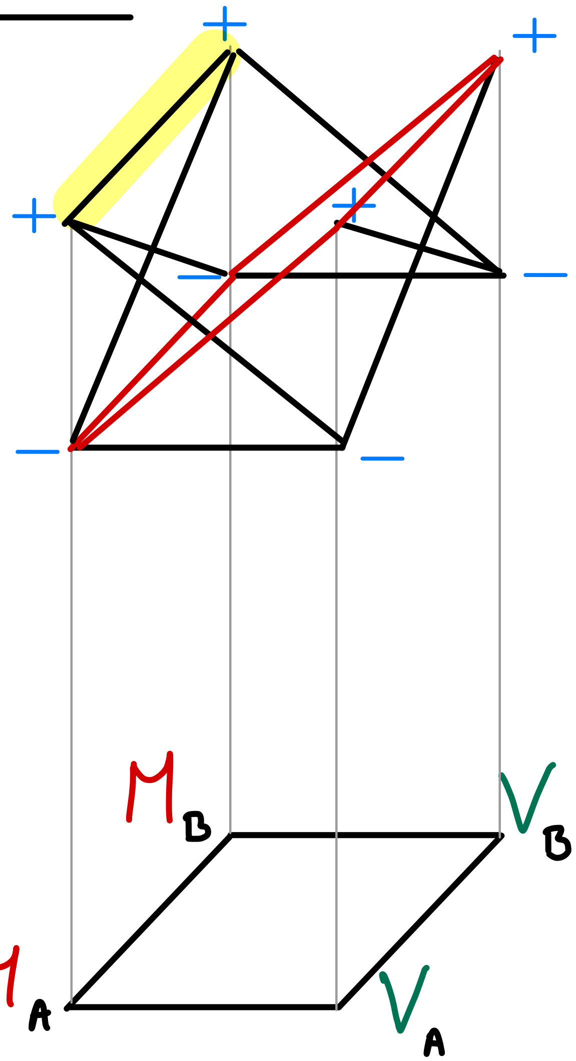


Global sections:

M_A	M_B	V_A	V_B
−	−	+	+

Local to Global

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	



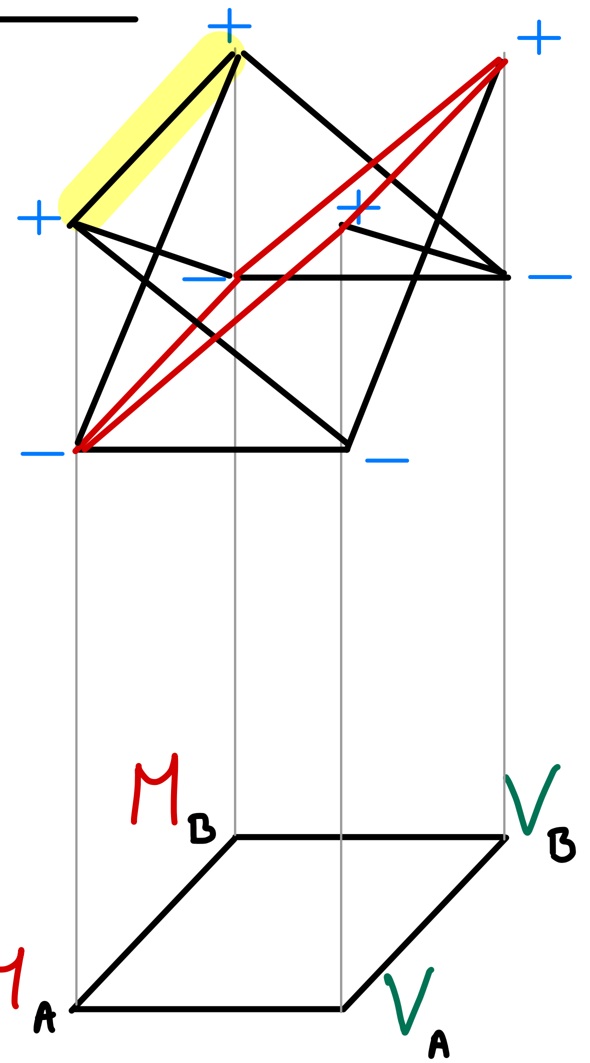
Global sections:

M_A	M_B	V_A	V_B
$-$	$-$	$+$	$+$

M_A	M_B	V_A	V_B
$+$	$+$	?	?

Local to Global

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	



Global sections:

M_A	M_B	V_A	V_B
−	−	+	+

" Logical Contextuality "

Hardy's Argument

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Hardy's Argument

Schmidt decomposition :

$$|\Psi\rangle = \alpha\, |a\rangle_1 |a\rangle_2 - \beta\, |b\rangle_1 |b\rangle_2$$

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Hardy's Argument

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Schmidt decomposition :

$$|\Psi\rangle = \alpha\, |a\rangle_1 |a\rangle_2 - \beta\, |b\rangle_1 |b\rangle_2$$

Chose observables

$$M_i = |m_i\rangle\langle m_i| \qquad V_i = |v_i\rangle\langle v_i|$$

Hardy's Argument

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Schmidt decomposition :

$$|\Psi\rangle = \alpha\,|a\rangle_1|b\rangle_1 - \beta\,|a\rangle_2|b\rangle_2$$

Chose observables

$$M_i = |m_i\rangle\langle m_i| \qquad V_i = |v_i\rangle\langle v_i|$$

where the + eigenvectors are

$$|m_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(\sqrt{\beta}\,|a\rangle_1 + \sqrt{\alpha}\,|a\rangle_2\right)$$

$$|v_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(\sqrt{\beta}\,|b\rangle_1 - \sqrt{\alpha}\,|b\rangle_2\right)$$

Hardy's Argument

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Schmidt decomposition :

$$|\Psi\rangle = \alpha\, |a\rangle_1 |b\rangle_1 - \beta\, |a\rangle_2 |b\rangle_2$$

Chose observables

$$M_i = |m_i\rangle\langle m_i| \qquad V_i = |v_i\rangle\langle v_i|$$

where the + eigenvectors are

$$|m_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\beta}\, |a\rangle_1 + \sqrt{\alpha}\, |a\rangle_2 \right)$$

$$|v_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\beta}\, |b\rangle_1 - \sqrt{\alpha}\, |b\rangle_2 \right)$$

and their orthogonal eigenvectors are

$$|n_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(-\sqrt{\alpha^*}\, |a\rangle_1 + \sqrt{\beta^*}\, |a\rangle_2 \right)$$

$$|u_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\alpha^*}\, |b\rangle_1 + \sqrt{\beta^*}\, |b\rangle_2 \right)$$

Hardy's Argument

	$+\ +$	$+\ -$	$-\ +$	$-\ -$
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

$$\langle \Psi | m_a \rangle | m_b \rangle \neq 0$$

Schmidt decomposition:

$$|\Psi\rangle = \alpha\, |a\rangle_1 |b\rangle_1 - \beta\, |a\rangle_2 |b\rangle_2$$

Chose observables

$$M_i = |m_i\rangle\langle m_i| \qquad V_i = |v_i\rangle\langle v_i|$$

where the + eigenvectors are

$$|m_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\beta}\, |a\rangle_1 + \sqrt{\alpha}\, |a\rangle_2 \right)$$

$$|v_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\beta}\, |b\rangle_1 - \sqrt{\alpha}\, |b\rangle_2 \right)$$

and their orthogonal eigenvectors are

$$|n_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(-\sqrt{\alpha^*}\, |a\rangle_1 + \sqrt{\beta^*}\, |a\rangle_2 \right)$$

$$|u_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}} \left(\sqrt{\alpha^*}\, |b\rangle_1 + \sqrt{\beta^*}\, |b\rangle_2 \right)$$

Hardy's Argument

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

$$\langle \Psi | m_a \rangle | m_b \rangle \neq 0$$

$$\langle \Psi | m_a \rangle | v_b \rangle = 0$$

Schmidt decomposition:

$$|\Psi\rangle = \alpha\, |a\rangle_1 |b\rangle_1 - \beta\, |a\rangle_2 |b\rangle_2$$

Chose observables

$$M_i = |m_i\rangle\langle m_i| \qquad V_i = |v_i\rangle\langle v_i|$$

where the + eigenvectors are

$$|m_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(\sqrt{\beta}\, |a\rangle_1 + \sqrt{\alpha}\, |a\rangle_2\right)$$

$$|v_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(\sqrt{\beta}\, |b\rangle_1 - \sqrt{\alpha}\, |b\rangle_2\right)$$

and their orthogonal eigenvectors are

$$|n_a\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(-\sqrt{\alpha^*}\, |a\rangle_1 + \sqrt{\beta^*}\, |a\rangle_2\right)$$

$$|u_b\rangle = \frac{1}{\sqrt{|\alpha| + |\beta|}}\left(\sqrt{\alpha^*}\, |b\rangle_1 + \sqrt{\beta^*}\, |b\rangle_2\right)$$

Our Argument

	+ +	+ −	− +	− −
M M	1	1	1	1
M V		1	1	1
V M		1	1	1
V V	1	1	1	

Just add one suitably chosen,
additional observable D_i
for each additional i^{th} qubit

Our Argument

	$+++$	$++-$	$+-+$	$+--$	$-++$	$-+-$	$--+$	$---$
D_1 M M	1	1	1	1	·	·	·	·
D_1 M V		1	1	1	·	·	·	·
D_1 V M		1	1	1	·	·	·	·
D_1 V V	1	1	1		·	·	·	·

Just add one suitably chosen,
additional observable D_i
for each additional i^{th} qubit

see paper for the exact formula (the "Going Up" lemmas)

ALGORITHM

Input	An n-qubit state $\vert\omega\rangle$
Output	Either
	Yes if $\vert\omega\rangle$ is logically contextual,
	together with a list of $n+2$ local observables, or
	No if $\vert\omega\rangle$ is in $\mathcal{P}_n$.

Base Cases

1. If $n = 1$, output No.
2. If $n = 2$, apply the Hardy procedure of the Base Case Lemma to the Schmidt decomposition of $|\omega\rangle$.

Recursive Case: $n+1$, $n > 1$

1. We apply $\mathsf{Test}\mathcal{P}_{n+1}$ to $|\omega\rangle$. If $|\omega\rangle$ is in $\mathcal{P}_{n+1}$, return No.
2. Otherwise, we write

$$|\omega\rangle \;=\; \alpha|\psi\rangle|0\rangle + \beta|\phi\rangle|1\rangle.$$

   Explicitly, if $|\omega\rangle$ is represented by a 2^{n+1}-dimensional complex vector

$$\sum_{\sigma\in\{0,1\}^{n+1}} a_\sigma|\sigma\rangle$$

   in the computational basis, we can define

$$\alpha = \sqrt{\sum_{\sigma\in\{0,1\}^n} |a_{\sigma 0}|^2}, \qquad \beta = \sqrt{\sum_{\sigma\in\{0,1\}^n} |a_{\sigma 1}|^2}$$

$$|\psi\rangle = \frac{1}{\alpha}\sum_{\sigma\in\{0,1\}^n} a_{\sigma 0}|\sigma\rangle, \qquad |\phi\rangle = \frac{1}{\beta}\sum_{\sigma\in\{0,1\}^n} a_{\sigma 1}|\sigma\rangle.$$

3. We apply $\mathsf{Test}\mathcal{P}_n$ to $|\psi\rangle$. If $|\psi\rangle$ is not in $\mathcal{P}_n$, we proceed recursively with $|\psi\rangle$, and then extend the observables using the construction of the Going Up Lemma I.
4. Otherwise, we proceed similarly with $|\phi\rangle$.

5. Otherwise, both $|\psi\rangle$ and $|\phi\rangle$ are in $\mathcal{P}_n$.
   For a in $(0,1)$, we define

$$\tau(a) \ := \ a|\psi\rangle + \sqrt{1-a^2}|\phi\rangle.$$

   For 19 distinct values in $(0,1)$, we assign these values to a, and apply $\mathsf{Test}\mathcal{P}_n$ to $\tau(a)$.
   If we find a value of a for which $\tau(a)$ is not in $\mathcal{P}_n$, we use that value to compute the local observable $B(\frac{\alpha}{a}, \frac{\beta}{\sqrt{1-a^2}})$ for the $n+1$-th party, as specified in the Going Up Lemma II, and continue the recursion with the n-qubit state $\tau(a)$.
6. Otherwise, by the 21 Lemma and the Small Difference Lemma, the only remaining case is where $|\psi\rangle$ and $|\phi\rangle$ differ in one qubit. We have these qubits $|\psi_1\rangle$, $|\phi_1\rangle$ from our previous applications of $\mathsf{Test}\mathcal{P}_n$. In this final case, we can write $|\omega\rangle$ as

$$|\omega\rangle = |\Psi\rangle \otimes |\xi\rangle$$

   where $|\Psi\rangle$ is in $\mathcal{P}_{n-1}$, and $|\xi\rangle$ is a 2-qubit state. Moreover, we have

$$|\xi\rangle = \alpha|\psi_1\rangle|0\rangle + \beta|\phi_1\rangle|1\rangle.$$

7. We apply the Base Case procedure to $|\xi\rangle$, which we know cannot be maximally entangled, by Step 1. We output Yes, together with the two local observables for each party produced by the Hardy construction, and the $n-2$ local observables for $|\Psi\rangle$ produced by the Corollary to the Going Up lemmas. □

SUBROUTINE $\mathsf{Test}\mathcal{P}_n$

Input n-qubit quantum state $|\theta\rangle$

Output Either

Yes, and entanglement type of $|\theta\rangle$, or

No

1. Compute the $n-1$ partial traces ρ_i over $n-1$ qubits of $|\theta\rangle$. If any ρ_i is not a maximally mixed state, compute $\mathsf{Tr}\rho_i^2$. If $\mathsf{Tr}\rho_i^2 \neq 1$, return No. We now have the list $\{i_1, \ldots, i_k\}$ of indices for which the maximally mixed state was returned.
2. For each i_p in the list, find its "partner" i_q by computing the partial traces ρ_{i_p,i_q} over $n-2$ qubits, and then testing if $\mathsf{Tr}\rho_{i_p,i_q}^2 = 1$.
   If we cannot find the partner for some i_p, return No.
3. Otherwise, we return Yes. We also have the complete entanglement type of $|\theta\rangle$, and we have computed all the single-qubit components. □