Variables, Pebbles, Width and Support

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Pebbling Co-Monad

(Abramsky, D., Wang 2017) introduced the *pebbling comonad* $\mathbb{P}_k$ on the category of σ-relational structures.

Morphisms $\mathbb{P}_k(\mathbb{A}) \longrightarrow \mathbb{B}$ describe *winning strategies* for Duplicator in a k-pebble one-sided game.

$\mathbb{P}_k(\mathbb{A}) \longrightarrow \mathbb{B}$ if, and only if, $\mathbb{A} \Rrightarrow^k \mathbb{B}$
where,

> $\mathbb{A} \Rrightarrow^k \mathbb{B}$ *denotes that every k-variable formula of existential positive first-order logic that is true in $\mathbb{A}$ is true in $\mathbb{B}$.*

Counting Logic

Isomorphisms $\mathbb{P}_k(\mathbb{A}) \cong \mathbb{P}_k(\mathbb{B})$ describe *winning strategies* for Duplicator in the k*-pebble bijection game*

$\mathbb{P}_k(\mathbb{A}) \cong \mathbb{P}_k(\mathbb{B})$ if, and only if, $\mathbb{A} \equiv^k \mathbb{B}$
where,

> $\mathbb{A} \equiv^k \mathbb{B}$ *denotes that every* k*-variable formula of first-order logic with counting that is true in* $\mathbb{A}$ *is true in* $\mathbb{B}$*.*

Why the focus on the number of variables?

Conjunctive Queries

Existential positive formulas are the closure under disjunctions of primitive positive formulas, also known as *conjunctive queries*.

Consider the query (in the language of *directed graphs*) saying *"there is a walk of length five"*.
In *prenex normal form* this requires *six* variables

$$\exists x_1 \cdots \exists x_6 (E(x_1, x_2) \land \cdots \land E(x_5, x_6)).$$

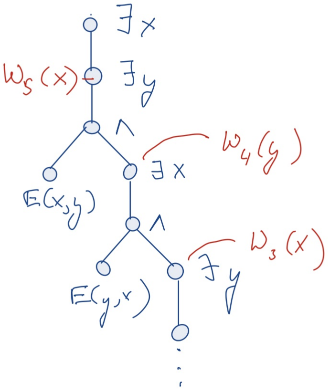
but, can be formulated with just *two*:

$$\exists x \exists y (E(x, y) \land \exists x (E(y, x) \land \exists y (\cdots))).$$

Query Plans

Formulating the query with a small number of variables allows for a *query plan* with small *intermediate relations*.

$$\exists x \exists y (E(x,y) \wedge \exists x (E(y,x) \wedge \exists y (\cdots))).$$



Tree Width

In general, for any structure $\mathbb{A}$, given a *tree decomposition* of $\mathbb{A}$ of width k, we can construct a *conjunctive query* $Q_{\mathbb{A}}$ with no more than $k+1$ variables such that

$\mathbb{B} \models Q_{\mathbb{A}}$ *if, and only if,* $\mathbb{A} \longrightarrow \mathbb{B}$.

(Kolaitis, Vardi)

In the pebbling comonad $\mathbb{P}_k$, from a *coalgebra* of $\mathbb{A}$, we can obtain a *tree decomposition* of $\mathbb{A}$ of width $k-1$.

Counting Logic

C^k is the k-variable fragment of first-order logic with *counting quantifiers*: $\exists^i x\theta$

Recall,
$\mathbb{P}_k(\mathbb{A}) \cong \mathbb{P}_k(\mathbb{B})$ if, and only if, $\mathbb{A} \equiv^k \mathbb{B}$
where, $\mathbb{A} \equiv^k \mathbb{B}$ denotes that the two structures agree on all sentences of C^k

The equivalence is characterised by the k-pebble *bijection game*.

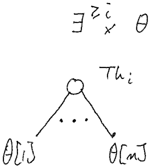
Circuits

Given a formula φ of C^k and $n \in \mathbb{N}$, we can define a *Boolean Circuit* with *threshold gates* of size $O(n^k)$ that takes as input a structure $\mathbb{A}$ on universe $\{1, \ldots, n\}$ and outputs True *iff* $\mathbb{A} \models \varphi$.

The inputs are labelled with *atomic facts*

$$R(a_1, \ldots, a_r) \quad a_1, \ldots, a_r \in \{1, \ldots, n\}$$

For every subformula $\theta(\vec{x})$ of φ and any assignment α of values from $\{1, \ldots, n\}$ to its free variables $\vec{x}$ we have a gate $\theta[\alpha]$.



Symmetric Circuits

If $\mathbb{A} \models \varphi$ for an n-element structure $\mathbb{A}$ then the circuit C_φ accepts the input for any mapping of the elements of $\mathbb{A}$ to the inputs $\{1, \ldots, n\}$.

The output of the circuit C_φ is *invariant* under permutations of $\{1, \ldots, n\}$.

Every permutation $\pi \in \mathrm{Sym}_n$ extends to an *automorphism* of C_φ.

We say that the circuit is Sym_n-*symmetric*.

From Circuits to Formulas

A family of Sym_n-symmetric circuits $(C_n)_{n \in \mathbb{N}}$, of size $O(n^k)$, taking as input σ-structures on n elements

can be transformed into a family $(\varphi_n)_{n \in \mathbb{N}}$ of formulas of C^{2k}

such that for any σ-structure $\mathbb{A}$ with n elements,

$\mathbb{A} \models \varphi_n$ *if, and only if,* C_n *accepts* $\mathbb{A}$.

Supports

Each gate g in C_n has an *invariance group*

$$\mathsf{Inv}_g = \{\pi \in \mathrm{Sym}_n \mid g(x^\pi) = g(x)\}$$

$$[\mathrm{Sym}_n : \mathsf{Inv}_g] \in O(n^k)$$

We can show that for each such gate, there is a *support*, i.e. a set $X \subseteq \{1, \ldots, n\}$ with $|X| \leq k$ such that

Any π that fixes X pointwise is in Inv_g.

Supports and Bijection Games

We can use *bijection games* and the *supports* to establish lower bounds for symmetric circuits.

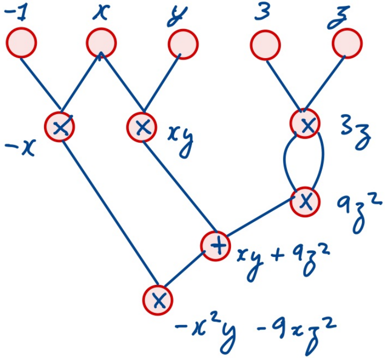
The key is the following connection.

> *If C is a symmetric circuit on n-vertex graphs such that every gate of C has a support of size at most k, and $\mathbb{A}$ and $\mathbb{B}$ are structures such that $\mathbb{A} \equiv^{2k} \mathbb{B}$ then:*
>
> *C accepts $\mathbb{A}$ if, and only if, C accepts $\mathbb{B}$.*

This can be proved by showing that if C distinguishes $\mathbb{A}$ from $\mathbb{B}$, then it provides a *winning strategy* for *Spoiler* in the $2k$-pebble bijection game.

Arithmetic Circuits

An *Arithmetic Circuit* over a field K computes (or represents) a *polynomial* in $K[X]$.



Matrix Inputs

We are often interested in inputs which are entries of *a matrix*.

$$X = \{x_{ij} \mid 1 \leq i \leq m; 1 \leq j \leq n\}$$

Especially, when the input is a *square matrix*, so $m = n$.

$$\det(X) = \sum_{\sigma \in \mathrm{Sym}_n} \mathsf{sgn}(\sigma) \prod_{i \in [n]} x_{i\sigma(i)}$$

$$\mathrm{per}(X) = \sum_{\sigma \in \mathrm{Sym}_n} \prod_{i \in [n]} x_{i\sigma(i)}$$

Valiant's conjecture VP $\neq$ VNP is that there are no *polynomial-size arithmetic circuits* for computing the *permanent*.

Symmetries of the Permanent

The *permanent*

$$\mathrm{per}(X) = \sum_{\sigma \in \mathrm{Sym}_n} \prod_{i \in [n]} x_{i\sigma(i)}$$

is invariant under *independent row and column permutations*.

That is, under the action of $\mathrm{Sym}_{[n]} \times \mathrm{Sym}_{[n]}$ given by

$$x_{ij}^{(\sigma,\pi)} = x_{\sigma(i)\pi(j)}.$$

We say that $\mathrm{per}(X)$ is *matrix symmetric*.

$\det(X)$ has fewer symmetries.

Determinant

The invariance group of

$$\det(X) = \sum_{\sigma \in \mathrm{Sym}_n} \mathsf{sgn}(\sigma) \prod_{i \in [n]} x_{i\sigma(i)}$$

includes

$$D = \{(\sigma, \pi) \in \mathrm{Sym}_{[n]} \times \mathrm{Sym}_{[n]} \mid \mathsf{sgn}(\sigma) = \mathsf{sgn}(\pi)\} \ltimes \mathbb{Z}_2.$$

In particular, it is $\mathrm{Alt}_{[n]} \times \mathrm{Alt}_{[n]}$-symmetric.

The defining expression yields a circuit with these symmetries, but of $\Omega(n!)$ size.

Results

From **(D., Wilsenach 2020 and 2022)**.

Γ	$\{\text{id}\}$	$\text{Sym}_{[n]}$	$\text{Alt}_{[n]} \times \text{Alt}_{[n]}$	$\text{Sym}_{[n]} \times \text{Sym}_{[n]}$
Det	$O(n^4)$	$O(n^4)$ *(char 0)*	$2^{\Omega(n)}$ *(char 0)*	N/A
Perm	$O(n^2 2^n)$ VP $=$ VNP?	$2^{\Omega(n)}$ *(char 0)*	$2^{\Omega(n)}$ *(char $\neq 2$)*	$2^{\Omega(n)}$ *(char $\neq 2$)*

Actually, all lower bounds are not just on the *size* of the circuit, but on *orbit size*.

Conclusion

The *grading* in the pebbling comonad is a fundamental resource that finds expression as:

- the number of *variables* in a formula;
- the number of *pebbles* in a model-comparison game;
- the *arity* of relations in a query plan;
- the *width* of tree decompositions of a structure;
- the *dimension* in Weisfeiler-Leman algorithms;
- the *support* of invariance groups in symmetric circuits.

Thank you!

28/03/2009, ETAPS