Lambek-Grishin Calculus:

Focusing, Display and Full Polarization

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Identity of proofs

Identifying or telling apart proofs has far-reaching consequences.

- Philosophy and math: when two proofs correspond to the same argument?
- Computer science: when two algorithms correspond to the same program?
- Linguistics: how to capture different readings of the same sentence?
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Natural deduction calculi or **proof nets**: less sensitive to rule permutations
$\rightsquigarrow$ **benchmarks** for defining identity of proofs.

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Focused sequent calculi [And92, And01, Mil04]: syntactic restrictions on rules:

1. the proof search space is reduced retaining **completeness**;
2. every cut-free proof comes in a **special normal form**;
3. criterion for defining **identity of sequent calculi proofs**.

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What is the mathematical underpinning of focalization?

Looking for:

- (uniform and modular structural) **proof theory** and
- (algebraic and categorical) **semantics**.

Our contributions

Lambek-Grishin logic:

- fully polarized algebraic semantics $\mathbb{FP.LG}$
- focused display calculus **fD.LG**:

- canonical cut-elimination, strong focalization, complete w.r.t. $\mathbb{FP.LG}$
- also complete w.r.t. LG-algebras

  $\rightsquigarrow$ **semantic proof of completeness of focusing**
- effective translation between **fD.LG**- and **fLG**-proofs [MM12]

  $\rightsquigarrow$ **operational semantics**

General theory:

- heterogenous display calculi
- fully polarized algebras
- analytic-inductive extensions

Basic Lambek-Grishin algebra

- Poset with 6 operations [Moo09]:

$$\begin{array}{ccccc} A \leq C \,/\, B & \text{iff} & A \otimes B \leq C & \text{iff} & B \leq A \backslash C \\ C \oslash B \leq A & \text{iff} & C \leq A \oplus B & \text{iff} & A \obslash C \leq B \end{array} \tag{1}$$

$$\frac{\text{John}}{np} \otimes \frac{\text{sleeps}}{np \backslash s} \overset{\text{is}}{\leq} \frac{\text{a sentence}}{s}$$

Multi-type Proper Display Calculi 1/2

Display calculi are a natural generalization of Gentzen's calculi [BJ82]

- **Display property**

  ⇝ semantically justified by **adjunction/residuation**

  If $\varepsilon_{f,i} = 1$ and $\varepsilon_{g,j} = 1$:

$$\hat{f} \dashv \check{f}_i^\sharp \ \begin{array}{c} \hat{f}\,(\overline{\Sigma})[\Gamma]_i \vdash \Delta \\ \hline\hline \Gamma \vdash \check{f}_i^\sharp\,(\overline{\Sigma})[\Delta]_i \end{array} \qquad \begin{array}{c} \Gamma \vdash \check{g}\,(\overline{\Sigma})[\Delta]_j \\ \hline\hline \hat{g}_j^\flat\,(\overline{\Sigma})[\Gamma]_j \vdash \Delta \end{array} \ \hat{g}_j^\flat \dashv \check{g}$$

  If $\varepsilon_{f,i} = \partial$ and $\varepsilon_{g,j} = \partial$:

$$(\hat{f}, \hat{f}_i^\sharp) \ \begin{array}{c} \hat{f}\,(\overline{\Sigma})[\Delta]_i \vdash \Delta' \\ \hline\hline \hat{f}_i^\sharp\,(\overline{\Sigma})[\Delta']_i \vdash \Delta \end{array} \qquad \begin{array}{c} \Gamma' \vdash \check{g}\,(\overline{\Sigma})[\Gamma]_j \\ \hline\hline \Gamma \vdash \check{g}_j^\flat\,(\overline{\Sigma})[\Gamma']_j \end{array} \ (\check{g}, \check{g}_j^\flat)$$

- **Canonical cut elimination**;

- **Properness:** all rules closed under **uniform substitution** [Wan98];

- **Multi-type:** **syntactic types** ⇜⇝ **subalgebras**

  ⇝ **uniform substitution within each type** [FGK+16].

Multi-type proper display calculi 2/2

Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper m.DC**.

Basic Lambek-Grishin logic

D.LG consists of the following rules (we consider only the Lambek fragment for brevity).

Axioms and cuts:

$$\frac{}{p \vdash p}\,\text{Id} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,\text{Cut}$$

Logical rules (i.e. **translation** vs **tonicity rules**, cfr. asynchronous vs synchronous [And01]):

$$\otimes_L\,\frac{A \mathbin{\hat{\otimes}} B \vdash X}{A \otimes B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \mathbin{\hat{\otimes}} Y \vdash A \otimes B}\,\otimes_R$$

$$\backslash_L\,\frac{X \vdash A \qquad B \vdash Y}{A \backslash B \vdash X \mathbin{\check{\backslash}} Y} \qquad \frac{X \vdash A \mathbin{\check{\backslash}} B}{X \vdash A \backslash B}\,\backslash_R \qquad /_L\,\frac{B \vdash Y \qquad X \vdash A}{B / A \vdash Y \mathbin{\check{/}} X} \qquad \frac{X \vdash B \mathbin{\check{/}} A}{X \vdash B / A}\,/_R$$

Display postulates:

$$\begin{array}{rc} & Y \vdash X \mathbin{\check{\backslash}} Z \\ \hat{\otimes} \dashv \check{\backslash} & \\ \hline\hline & X \mathbin{\hat{\otimes}} Y \vdash Z \\ \hat{\otimes} \dashv \check{/} & \\ \hline\hline & X \vdash Z \mathbin{\check{/}} Y \end{array}$$

We may expand the calculus with so-called **Structural rules**, e.g.:

$$\begin{array}{c} (X \mathbin{\hat{\otimes}} Y) \mathbin{\hat{\otimes}} Z \vdash W \\ \hline\hline X \mathbin{\hat{\otimes}} (Y \mathbin{\hat{\otimes}} Z) \vdash W \end{array}\; a, a^{-1}$$

Everybody needs somebody

$$\frac{\text{Everybody}}{s\,/\,(np\,\backslash\,s)} \otimes \frac{\text{needs}}{(((np\,\backslash\,s)\,/\,((s\,/\,np)\,\backslash\,s))} \otimes \frac{\text{somebody}}{(s\,/\,np)\,\backslash\,s} \overset{\text{is}}{\vdash} \frac{\text{a sentence}}{\text{s}}$$

7 sequent derivations, but only 3 ND (or proof net) derivations in normal form!

Moving to a focused sequent system (**fLG** or **fD.LG**) again 3 derivations!

Two derivations use associativity and correspond to the following readings:

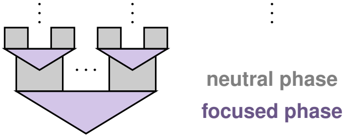
- ∀-∃ reading: Everybody > somebody > needs
- ∃-∀ reading: Somebody > everybody > needs

Focalization

The key idea relies on the following distinction.

- A **focused phase** is a proof-section where a formula is decomposed "as much as possible" only by means of **tonicity logical rules**.
- A **neutral phase** is a non-focused phase, i.e. a proof section built by **translation logical rules** (applied greedily) or **structural rules**.

A **strongly focalized proof** exhibits a strict alternation between focused and neutral phases:



Definition

A sequent proof π is **strongly focalized** if cut-free and, for every formula A occurring in π, every **PIA subtree** of A is constructed by a proof-section of π containing only **tonicity rules**.

Focalization via polarization

Two focalized phases

- **Positive phase**: only tonicity rules for positive connectives are applied;
- **Negative phase**: only tonicity rules for negative connectives are applied.

How to categorize a connective as "positive" or "negative"?

- **Positive formulas**: the main connective is a **left-adjoint/residual** $(\oslash, \otimes, \obslash)$;
- **Negative formulas**: the main connective is a **right-adjoint/residual** $(/ , \oplus, \backslash)$.

The key idea of polarization "naturally" calls for a type distinction:

multi-type calculi seem good candidates, but we need a further generalization!

A step back: focalization via "implicit" polarization

State of the art: **fLG** by Moortgat and Moot [MM12]

- Every proof is strongly focalized
- Focus implemented by a meta-linguistical marker $\boxed{A}$
- **Restrictions on the applicability of rules**

If A is positive:

Axiom	Focusing	Defocusing
$\dfrac{}{A \vdash \boxed{A}}$	$\leftharpoondown \dfrac{A \vdash \Delta}{\boxed{A} \vdash \Delta}$	$\dfrac{X \vdash \boxed{A}}{X \vdash A} \rightharpoonup$

If A is negative:

Co-axiom	Focusing	Defocusing
$\dfrac{}{\boxed{A} \vdash A}$	$\dfrac{X \vdash A}{X \vdash \boxed{A}} \rightharpoondown$	$\leftharpoonup \dfrac{\boxed{A} \vdash X}{A \vdash X}$

Tonicity rules have auxiliary and principal formulas in focus.

∀-∃ reading: **fLG**

Bias assignment: *np* :: positive, *s* :: negative.

$$\dfrac{\dfrac{\dfrac{}{\boxed{s_0} \vdash s_{11}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{np_1 \vdash \boxed{np_3}} \quad \dfrac{}{\boxed{s_4} \vdash s_8}}{\boxed{np \backslash s} \vdash np \mathbin{\check{\backslash}} s} \quad \dfrac{\dfrac{\dfrac{}{\boxed{s_5} \vdash s_7} \quad \dfrac{}{np_9 \vdash \boxed{np_6}}}{\boxed{s / np} \vdash s \mathbin{\check{/}} np}}{np \vdash \boxed{(s / np) \backslash s}}\,\leftrightharpoons}{\boxed{(np \backslash s) / ((s / np) \backslash s)} \vdash (np \mathbin{\check{\backslash}} s) \mathbin{\check{/}} np}}{np \mathbin{\hat{\otimes}} ((np \backslash s) / ((s / np) \backslash s)) \vdash \boxed{s / np}}\,\leftrightharpoons, a \quad \dfrac{}{\boxed{s_{10}} \vdash s_2}}{\boxed{(s_8 / np_9) \backslash s_{10}} \vdash (np \mathbin{\hat{\otimes}} ((np \backslash s) / (s / np) \backslash s))) \mathbin{\check{\backslash}} s}}{((np \backslash s) / ((s / np) \backslash s)) \mathbin{\hat{\otimes}} ((s / np) \backslash s) \vdash \boxed{np \backslash s}}\,\leftrightharpoons, a^{-1}}{\boxed{s_0 / (np_1 \backslash s_2)} \vdash s \mathbin{\check{/}} (((np \backslash s) / ((s / np) \backslash s)) \mathbin{\hat{\otimes}} ((s / np) \backslash s))}}{\underbrace{s_0 / (np_1 \backslash s_2)}_{\textit{Everybody}} \mathbin{\hat{\otimes}} (\underbrace{((np_3 \backslash s_4) / ((s_5 / np_6) \backslash s_7))}_{\textit{needs}} \mathbin{\hat{\otimes}} \underbrace{((s_8 / np_9) \backslash s_{10})}_{\textit{somebody}}) \vdash \boxed{s_{11}}}\,\leftrightharpoons$$

∃-∀ reading: **fLG**

Bias assignment: *np* :: positive, *s* :: negative.

$$\dfrac{\dfrac{}{\boxed{s_5} \vdash s_7} \qquad \dfrac{}{np_9 \vdash \boxed{np_6}}}{\dfrac{\boxed{s \,/\, np} \vdash s \mathbin{\check{/}} np}{np \vdash \boxed{(s \,/\, np) \backslash s}}\ \leftrightharpoons}$$

$$\dfrac{\dfrac{}{np_1 \vdash \boxed{np_3}} \qquad \dfrac{}{\boxed{s_4} \vdash s_2}}{\boxed{np \backslash s} \vdash np \mathbin{\check{\backslash}} s}$$

$$\dfrac{\boxed{np \backslash s} \vdash np \mathbin{\check{\backslash}} s \qquad np \vdash \boxed{(s \,/\, np) \backslash s}}{\dfrac{\boxed{(np \backslash s) \,/\, ((s \,/\, np) \backslash s)} \vdash (np \mathbin{\check{\backslash}} s) \mathbin{\check{/}} np}{((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \mathbin{\hat{\otimes}} np \vdash \boxed{np \backslash s}}\ \leftrightharpoons}$$

$$\dfrac{\dfrac{}{\boxed{s_0} \vdash s_8} \qquad ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \mathbin{\hat{\otimes}} np \vdash \boxed{np \backslash s}}{\dfrac{\boxed{s_0 \,/\, (np_1 \backslash s_2)} \vdash s \mathbin{\check{/}} (((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \mathbin{\hat{\otimes}} np)}{(s_0 \,/\, (np_1 \backslash s_2)) \mathbin{\hat{\otimes}} ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \vdash \boxed{s \,/\, np}}\ \leftrightharpoons, a}$$

$$\dfrac{(s_0 \,/\, (np_1 \backslash s_2)) \mathbin{\hat{\otimes}} ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \vdash \boxed{s \,/\, np} \qquad \dfrac{}{\boxed{s_{10}} \vdash s_{11}}}{\dfrac{\boxed{(s_8 \,/\, np_9) \backslash s_{10}} \vdash ((s \,/\, (np \backslash s)) \mathbin{\hat{\otimes}} ((np \backslash s) \,/\, ((s \,/\, np) \backslash s))) \mathbin{\check{\backslash}} s}{\underbrace{s_0 \,/\, (np_1 \backslash s_2)}_{\textit{Everybody}} \mathbin{\hat{\otimes}} (\underbrace{((np_3 \backslash s_4) \,/\, ((s_5 \,/\, np_6) \backslash s_7))}_{\textit{needs}} \mathbin{\hat{\otimes}} \underbrace{((s_8 \,/\, np_9) \backslash s_{10})}_{\textit{somebody}}) \vdash \boxed{s_{11}}}\ \leftrightharpoons, a^{-1}}$$

Focalization via "explicit" polarization

In proof-theory, **shift operators** have been considered:

- if A is negative, then $\downarrow A$ is positive;
- if A is positive, then $\uparrow A$ is negative.

$\rightsquigarrow$ but their status as operators is obscure.

In algebraic/categorical polarized semantics:

- $\uparrow \dashv \downarrow$
- $\uparrow\downarrow\uparrow \varphi = \uparrow \varphi$
- $\downarrow\uparrow\downarrow \varphi = \downarrow \varphi$
- $\downarrow\uparrow \varphi = \varphi$
- $\uparrow\downarrow \varphi = \varphi$.

Problem: the focusing policy could be destroyed.
$\rightsquigarrow$ the usual solution is to consider only sequents where $\uparrow$ (resp. $\downarrow$) are not nested.

Our solution: 4 types:

- **positive** and **negative** formulas belong to different sorts;
- **pure** and **shifted** formulas belong to different sorts.

Weakening relations

W.R. are the **order-theoretic equivalents** of **profunctors** (aka distributors or bimodules) [Ben73].

W.R. are **generalizations** of **partial orders**: take $\mathcal{A} = \mathcal{B}$ and $\leq_{\mathcal{A}} = \leq_{\mathcal{B}}$.

Definition

A **weakening relation** is a relation $\preccurlyeq \subseteq \mathcal{A} \times \mathcal{B}$ on two partially ordered set $(\mathcal{A}, \leq_{\mathcal{A}})$ and $(\mathcal{B}, \leq_{\mathcal{B}})$ that is **compatible with the orders** $\leq_{\mathcal{A}}$ and $\leq_{\mathcal{B}}$ in the following sense

$$\frac{A' \leq_{\mathcal{A}} A \qquad A \preccurlyeq B \qquad B \leq_{\mathcal{B}} B'}{A' \preccurlyeq B'}$$

Definition

Given two w.r. $\preccurlyeq_{\mathcal{A}} \subseteq \mathcal{A} \times \mathcal{A}'$ and $\preccurlyeq_{\mathcal{B}} \subseteq \mathcal{B} \times \mathcal{B}'$, we say that the order-preserving functions $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B}' \to \mathcal{A}'$ form a **heterogeneous adjoint pair** $L \dashv^{\preccurlyeq_{\mathcal{B}}}_{\preccurlyeq_{\mathcal{A}}} R$ if for every $A \in \mathcal{A}$ and $B' \in \mathcal{B}'$,

$$L(A) \preccurlyeq_{\mathcal{B}} B' \text{ iff } A \preccurlyeq_{\mathcal{A}} R(B')$$

$$\begin{tikzcd}
\mathcal{A}' & & \mathcal{B}' \arrow[ll, bend right, "R"'] \\
\mathcal{A} \arrow[u, "\preccurlyeq_{\mathcal{A}}", "|" marking] \arrow[rr, bend right, "L"'] \arrow[urr, phantom, "\top"] & & \mathcal{B} \arrow[u, "\preccurlyeq_{\mathcal{B}}"', "|" marking]
\end{tikzcd}$$

If $\mathcal{A}' = \mathcal{A}$, $\preccurlyeq_{\mathcal{A}} = \leq_{\mathcal{A}}$, $\mathcal{B}' = \mathcal{B}$ and $\preccurlyeq_{\mathcal{B}} = \leq_{\mathcal{B}}$, we recover the usual definition of adjunction.

Heterogeneous adjunctions also appear in the theory of Chu spaces.

Full polarization

$$\begin{tikzcd}
\dot{\mathbb{P}} \arrow[loop left, "\shortmid" marking, "{\cdot\!\preceq\!\cdot}^{+}"] \arrow[rr, "\shortmid" marking, "\cdot\!\preceq"] & & \mathbb{N} \arrow[loop right, "\shortmid" marking, "\preceq^{-}"] \\
\\
\mathbb{P} \arrow[loop left, "\shortmid" marking, "\preceq^{+}"] \arrow[uu, "\shortmid" marking, "{\preceq\!\cdot}^{+}"] \arrow[uurr, "\shortmid" marking, "\preceq"] \arrow[rr, "\shortmid" marking, "\preceq\!\cdot"'] & & \dot{\mathbb{N}} \arrow[loop right, "\shortmid" marking, "{\cdot\!\preceq\!\cdot}^{-}"] \arrow[uu, "\shortmid" marking, "{\cdot\!\preceq}^{-}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathring{\mathbb{P}} \arrow[loop left, "\shortmid" marking, "\mathring{\preceq}^{+}"] \arrow[rr, "\shortmid" marking, "\mathring{\preceq}"] & & \mathring{\mathbb{N}} \arrow[loop right, "\shortmid" marking, "\mathring{\preceq}^{-}"]
\end{tikzcd}$$

Fully polarized LG-algebras $\mathbb{FP}.\mathbb{LG}$

Operators and their residuals (we consider the Lambek fragment for brevity):

$$/ \ : \ \mathring{\mathbb{N}} \times \mathring{\mathbb{P}}^{\partial} \to \mathbb{N}, \quad \otimes : \ \mathring{\mathbb{P}} \times \mathring{\mathbb{P}} \to \mathbb{P}, \quad \backslash \ : \ \mathring{\mathbb{P}}^{\partial} \times \mathring{\mathbb{N}} \to \mathbb{N}.$$

$$\mathring{P} \mathring{\leq} \mathring{N} / \mathring{Q} \quad \text{iff} \quad \mathring{P} \otimes \mathring{Q} \mathring{\leq} \mathring{N} \quad \text{iff} \quad \mathring{Q} \mathring{\leq} \mathring{P} \backslash \mathring{N}. \tag{2}$$

Shifts:

$$\begin{tikzcd} \mathbb{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \dot{\mathbb{N}} \arrow[l, bend left, "\downharpoonright"] \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathbb{P}} \arrow[r, bend left, "\upharpoonleft"] \arrow[r, phantom, "\perp"] & \mathbb{N} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{3}$$

For all $P \in \mathbb{P}$ and $N \in \mathbb{N}$, $\leq\cdot^{+} \ \subseteq \mathbb{P} \times \dot{\mathbb{P}}$, $\leq \ \subseteq \mathbb{P} \times \mathbb{N}$ and $\cdot\leq^{-} \ \subseteq \dot{\mathbb{N}} \times \mathbb{N}$ are s.t.:

$$\uparrow P \cdot\leq^{-} N \quad \text{iff} \quad P \leq N \quad \text{iff} \quad P \leq\cdot^{+} \downarrow N \tag{4}$$

i.e. $\leq$ is the **weakening relation** represented by the **heterogeneous adjunction** $\uparrow \dashv^{\leq\cdot^{+}}_{\cdot\leq^{-}} \downarrow$.

Collage posets: $(\mathring{\mathbb{P}}, \mathring{\leq}^{+}) := (\mathbb{P} \sqcup \dot{\mathbb{P}}, \leq^{+} \sqcup \leq\cdot^{+} \sqcup \cdot\leq\cdot^{+})$, $(\mathring{\mathbb{N}}, \mathring{\leq}^{-}) := (\mathbb{N} \sqcup \dot{\mathbb{N}}, \leq^{-} \sqcup \cdot\leq^{-} \sqcup \cdot\leq\cdot^{-})$.

Collage weakening relation: $\mathring{\leq} := \leq\cdot \sqcup \leq \sqcup \cdot\leq \ \subseteq \ \mathring{\mathbb{P}} \times \mathring{\mathbb{N}}$.

(Heterogeneous multi-type proper) focused display calculus **fD.LG**

Notation: $\mathring{P} \in \{P, \dot{P}\}$, resp. $\mathring{N} \in \{N, \dot{N}\}$.

$$
\begin{array}{lcll}
P & := & p \mid \mathring{P} \otimes \mathring{P} \mid (\mathring{P} \oslash \mathring{N}) \mid (\mathring{N} \obslash \mathring{P}) & \text{Pure positive formulas } (\check{\downarrow}\, \dot{\Delta}) \\
N & := & n \mid (\mathring{N} \oplus \mathring{N}) \mid \mathring{P} \backslash \mathring{N} \mid \mathring{N} / \mathring{P} & \text{Pure negative formulas } (\hat{\uparrow}\, \dot{X}) \\
\dot{P} & := & \downarrow N & \text{Shifted positive formulas} \\
\dot{N} & := & \uparrow P & \text{Shifted negative formulas}
\end{array}
$$

Well-formed sequents (sequents in grey cells are not derivable):

Positive sequents	$X \vdash^{+} Y$	$\dot{X} \cdot\!\vdash^{+} Y$	$X \vdash\!\cdot^{+} \dot{Y}$	$\dot{X} \cdot\!\vdash\!\cdot^{+} \dot{Y}$
Negative sequents	$\Delta \vdash^{-} \Gamma$	$\dot{\Delta} \cdot\!\vdash^{-} \Gamma$	$\Delta \vdash\!\cdot^{-} \dot{\Gamma}$	$\dot{\Delta} \cdot\!\vdash\!\cdot^{-} \dot{\Gamma}$
Neutral sequents	$X \vdash \Delta$	$\dot{X} \cdot\!\vdash \Delta$	$X \vdash\!\cdot \dot{\Delta}$	$\dot{X} \cdot\!\vdash\!\cdot \dot{\Delta}$

(5)

Each consequence relation is interpreted by a W.R. as follows:

t	$\vdash^{+}$	$\vdash\!\cdot^{+}$	$\cdot\!\vdash\!\cdot^{+}$	$\vdash^{-}$	$\cdot\!\vdash^{-}$	$\cdot\!\vdash\!\cdot^{-}$	$\vdash$	$\cdot\!\vdash$	$\vdash\!\cdot$	$\mathring{\vdash}^{+}$	$\mathring{\vdash}^{-}$	$\mathring{\vdash}$
$t^{\mathbb{FP}.\mathbb{LG}}$	$\leq^{+}$	$\preceq\!\cdot^{+}$	$\cdot\!\leq\!\cdot^{+}$	$\leq^{-}$	$\cdot\!\preceq^{-}$	$\cdot\!\leq\!\cdot^{-}$	$\preceq$	$\cdot\!\preceq$	$\preceq\!\cdot$	$\mathring{\leq}^{+}$	$\mathring{\leq}^{-}$	$\mathring{\preceq}$

(6)

Axioms and cuts

$$\begin{gathered}
\frac{}{p \vdash^{+} p}\,p\text{-Id} \qquad n\text{-Id}\,\frac{}{n \vdash^{-} n} \\[2ex]
\text{P-Cut}\,\frac{\mathring{X} \overset{\circ}{\vdash}^{+} \mathring{P} \qquad \mathring{P} \overset{\circ}{\vdash}^{+} \mathring{Y}}{\mathring{X} \overset{\circ}{\vdash}^{+} \mathring{Y}} \qquad \frac{\mathring{\Gamma} \overset{\circ}{\vdash}^{-} \mathring{N} \qquad \mathring{N} \overset{\circ}{\vdash}^{-} \mathring{\Delta}}{\mathring{\Gamma} \overset{\circ}{\vdash}^{-} \mathring{\Delta}}\,\text{N-Cut} \\[2ex]
\text{Pn-Cut}\,\frac{\mathring{X} \overset{\circ}{\vdash}^{+} \mathring{P} \qquad \mathring{P} \overset{\circ}{\vdash} \mathring{\Delta}}{\mathring{X} \overset{\circ}{\vdash} \mathring{\Delta}} \qquad \frac{\mathring{X} \overset{\circ}{\vdash} \mathring{N} \qquad \mathring{N} \overset{\circ}{\vdash}^{-} \mathring{\Delta}}{\mathring{X} \overset{\circ}{\vdash} \mathring{\Delta}}\,\text{nN-Cut}
\end{gathered} \tag{7}$$

Logical rules

$$\otimes_L\ \frac{\mathring{P}\,\hat{\otimes}\,\mathring{Q}\ \overset{\circ}{\vdash}\ \mathring{\Delta}}{\mathring{P}\otimes\mathring{Q}\ \overset{\circ}{\vdash}\ \mathring{\Delta}} \qquad \frac{\mathring{X}\ \overset{\circ}{\vdash}{}^{+}\ \mathring{P} \qquad \mathring{Y}\ \overset{\circ}{\vdash}{}^{+}\ \mathring{Q}}{\mathring{X}\,\hat{\otimes}\,\mathring{Y}\ \vdash^{+}\ \mathring{P}\otimes\mathring{Q}}\ \otimes_R$$

$$\backslash_L\ \frac{\mathring{X}\ \overset{\circ}{\vdash}{}^{+}\ \mathring{P} \qquad \mathring{N}\ \overset{\circ}{\vdash}{}^{-}\ \mathring{\Delta}}{\mathring{P}\,\backslash\,\mathring{N}\ \vdash^{-}\ X\,\check{\backslash}\,\Delta} \qquad \frac{\mathring{X}\ \overset{\circ}{\vdash}\ \mathring{P}\,\check{\backslash}\,\mathring{N}}{\mathring{X}\ \overset{\circ}{\vdash}\ \mathring{P}\,\backslash\,\mathring{N}}\ \backslash_R \qquad /_L\ \frac{\mathring{N}\ \overset{\circ}{\vdash}{}^{-}\ \mathring{\Delta} \qquad \mathring{X}\ \overset{\circ}{\vdash}{}^{+}\ \mathring{P}}{\mathring{N}\,/\,\mathring{P}\ \vdash^{-}\ \mathring{\Delta}\,\check{/}\,\mathring{X}} \qquad \frac{\mathring{X}\ \overset{\circ}{\vdash}\ \mathring{N}\,\check{/}\,\mathring{P}}{\mathring{X}\ \overset{\circ}{\vdash}\ \mathring{N}\,/\,\mathring{P}}\ /_R$$

$$\downarrow_L\ \frac{N\ \vdash^{-}\ \Delta}{\downarrow N\ \cdot\!\vdash\!\cdot^{+}\ \check{\downarrow}\,\Delta} \qquad \frac{\mathring{X}\ \overset{\circ}{\vdash}{}^{+}\ \check{\downarrow}N}{\mathring{X}\ \overset{\circ}{\vdash}{}^{+}\ \downarrow N}\ \downarrow_R \qquad \uparrow_L\ \frac{\hat{\uparrow}P\ \overset{\circ}{\vdash}{}^{-}\ \mathring{\Delta}}{\uparrow P\ \overset{\circ}{\vdash}{}^{-}\ \mathring{\Delta}} \qquad \frac{X\ \vdash^{+}\ P}{\hat{\uparrow}X\ \cdot\!\vdash\!\cdot^{-}\ \uparrow P}\ \uparrow_R \tag{8}$$

Structural rules

Display postulates:

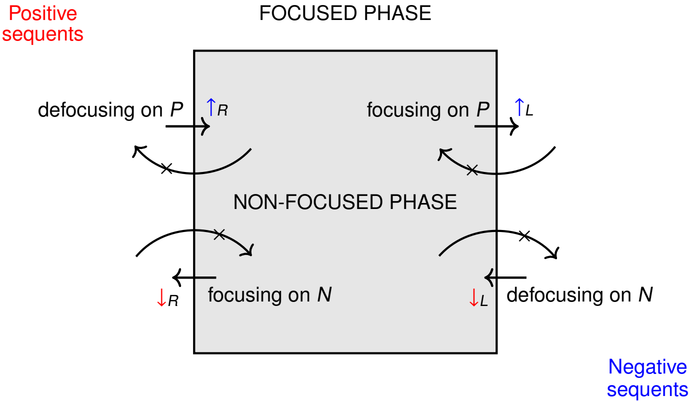
$$\begin{array}{c} \hat{\otimes} \dashv \check{\backslash} \\[2em] \hat{\otimes} \dashv \check{/} \end{array} \begin{array}{c} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{X} \mathbin{\check{\backslash}} \mathring{\Delta} \\ \hline\hline \mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta} \mathbin{\check{/}} \mathring{Y} \end{array} \qquad \begin{array}{c} \mathring{X} \mathbin{\hat{\oslash}} \mathring{\Delta} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \\ \hline\hline \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \mathbin{\check{\oplus}} \mathring{\Delta} \\ \hline\hline \mathring{\Gamma} \mathbin{\hat{\obslash}} \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta} \end{array} \begin{array}{c} \hat{\oslash} \dashv \check{\oplus} \\[2em] \hat{\obslash} \dashv \check{\oplus} \end{array}$$

$$\begin{array}{c} \hat{\uparrow} X \mathrel{\cdot\vdash\cdot^{-}} \dot{\Delta} \\ \hline\hline X \vdash^{+} \check{\downarrow} \dot{\Delta} \end{array}\ \hat{\uparrow} \dashv \check{\downarrow} \qquad \begin{array}{c} \hat{\uparrow} X \mathrel{\cdot\vdash^{-}} \Delta \\ \hline\hline X \mathrel{\vdash\cdot^{+}} \check{\downarrow} \Delta \end{array}\ \hat{\uparrow} \dashv \check{\downarrow} \qquad \hat{\uparrow} \dashv \check{\downarrow}\ \begin{array}{c} \dot{X} \mathrel{\cdot\vdash\cdot^{+}} \check{\downarrow} \Delta \\ \hline\hline \hat{\uparrow} \dot{X} \vdash^{-} \Delta \end{array} \tag{9}$$

Structural rules for shifts:

$$\begin{array}{c} \mathring{X} \mathrel{\mathring{\vdash}} \Delta \\ \hline\hline \mathring{X} \mathrel{\mathring{\vdash}^{+}} \check{\downarrow} \Delta \end{array}\ \check{\downarrow} \qquad \hat{\uparrow}\ \begin{array}{c} X \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline \hat{\uparrow} X \mathrel{\mathring{\vdash}^{-}} \mathring{\Delta} \end{array} \tag{10}$$

Phases and phase transitions 1/2



Positive sequents

FOCUSED PHASE

defocusing on P $\uparrow_R$

focusing on P $\uparrow_L$

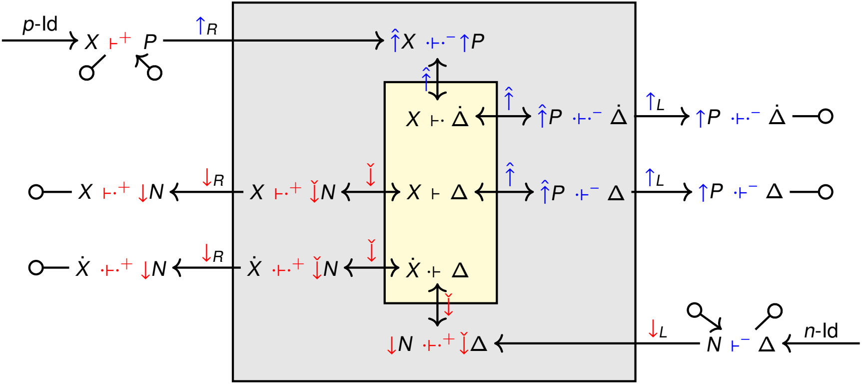
NON-FOCUSED PHASE

$\downarrow_R$ focusing on N

$\downarrow_L$ defocusing on N

Negative sequents

Phases and phase transitions 2/2



∀-∃ reading: **fD.LG**

Bias assignment: *np* :: positive, *s* :: negative.

$$
\dfrac{
\dfrac{}{s_0 \vdash^{-} s_{11}}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{np_1 \vdash^{+} np_3} \quad \dfrac{}{s_4 \vdash^{-} s_8}
}{np \backslash s \vdash^{-} np \,\check{\backslash}\, s}
\quad
\dfrac{
\dfrac{
\dfrac{}{s_5 \vdash^{-} s_7} \quad \dfrac{}{np_9 \vdash^{+} np_6}
}{s \,/\, np \vdash^{-} s \,\check{/}\, np}
}{np \vdash\!\cdot^{+} \downarrow(\downarrow(s \,/\, np) \backslash s)} \leftrightharpoons
}{(np \backslash s) \,/\, \downarrow(\downarrow(s \,/\, np) \backslash s) \vdash^{-} (np \,\check{\backslash}\, s) \,\check{/}\, np}
}{np \,\hat{\otimes} \downarrow((np \backslash s) \,/\, \downarrow(\downarrow(s \,/\, np) \backslash s)) \vdash\!\cdot^{+} \downarrow(s \,/\, np)} \leftrightharpoons, a
\quad
\dfrac{}{s_{10} \vdash^{-} s_2}
}{\downarrow(s_8 \,/\, np_9) \backslash s_{10} \vdash^{-} (np \,\hat{\otimes} \downarrow((np \backslash s) \,/\, \downarrow(\downarrow(s \,/\, np) \backslash s))) \,\check{\backslash}\, s}
}{\downarrow((np \backslash s) \,/\, \downarrow(\downarrow(s \,/\, np) \backslash s)) \,\hat{\otimes} \downarrow(\downarrow(s \,/\, np) \backslash s) \vdash\!\cdot^{+} \downarrow(np \backslash s)} \leftrightharpoons, a^{-1}
}{
\dfrac{s_0 \,/\, \downarrow(np_1 \backslash s_2) \vdash^{-} s \,\check{/}\, (\downarrow((np \backslash s) \,/\, \downarrow(\downarrow(s \,/\, np) \backslash s)) \,\hat{\otimes} \downarrow(\downarrow(s \,/\, np) \backslash s))}{\underbrace{\downarrow(s_0 \,/\, \downarrow(np_1 \backslash s_2))}_{\textit{Everybody}} \,\hat{\otimes}\, (\underbrace{\downarrow((np_3 \backslash s_4) \,/\, \downarrow(\downarrow(s_5 \,/\, np_6) \backslash s_7))}_{\textit{needs}} \,\hat{\otimes}\, \underbrace{\downarrow(\downarrow(s_8 \,/\, np_9) \backslash s_{10})}_{\textit{somebody}}) \vdash s_{11}} \leftrightharpoons
}
$$

Conclusions

What we did.

Lambek-Grishin logic:

- focused display calculus **fD.LG**

  ⇝ canonical cut-elimination and strong focalization
- fully polarized algebraic semantics $\mathbb{FP}.\mathbb{LG}$

  ⇝ semantic proof of completeness of focusing

Future work.

We expect that the present approach:

(1) extends to every (first-order) displayable logic;

(2) can be lifted to categories (profunctors instead of w.r.).

(2.a) So far, we defined a **categorical Lindenbaum-Tarsky construction** producing the free **residuated category** generated by **D.L**;

(2.b) we defined an algorithm transfoming

sequent derivations ⇝ **diagrams** ⇝ **natural deduction derivations.**

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