“It’s all Greek to me”

– on the pre-history of categorical logic

Peter M. Hines — University of York

U.C.L. , Sept. 2023

A puzzle from antiquity . . .

The Logician/Philosopher vs. the Astronomer/Mathematician :

"Chrysippus says that the number of conjunctions[a] [constructible] from only ten assertibles exceeds one hundred myriads [i.e. 10^6]. However, Hipparchus refuted this, demonstrating that the affirmative encompasses 103049 conjoined assertibles and the negative 310952."

— *Plutarch,* **Quæstiones Convivales** *(2nd C. AD)*

[a] 'combinations' in some documents . . .

This was reported as *common knowledge*,

"Chrysippus is refuted by *all the arithmeticians*, among them Hipparchus himself who proves that his error in calculation is enormous".

— *Plutarch,* **De Stoicorum Repugnantiis** *(2nd C. AD)*

but the precise meaning was lost.

"Since the exact terms of the problem are not stated, it is difficult to interpret the numerical answers . . . The Greeks took no interest in these matters".

— *N. L. Biggs* **The Roots of Combinatorics** *1979*

The significance of 103049 was realised in 1994 by Daniel Hough :

> Hipparchus, Plutarch, Schröder, and Hough
> — *R. Stanley, American Mathematical Monthly (1997)*

It is simply the 10^{th} **little Schröder number**, counting (amongst other things[1]) the number of distinct **Rooted Planar Trees** with ten leaves.

A natural (too easy?) interpretation

We may view :

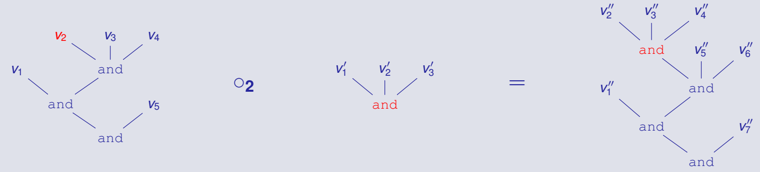
1. Each **branching** as a logical operation (conjunction?)
2. Each **leaf** as a simple assertible (variable?)

Building larger trees from smaller trees : **Substituting a tree for a given leaf.**

[1] e.g. the number of facets of the tenth associahedron

Replacing simple assertibles by non-simple composites

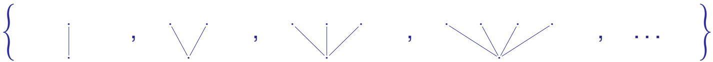
Operadic Composition :



Provided we

- Avoid clashes of free variable names, & identify α-equivalent trees,
- Identify up to (half-planar) topological equivalence,

we arrive at the non-symmetric operad $\mathbb{RPT}$ of **rooted planar trees**.
This is *freely generated* by one tree of each arity (number of leaves).



Counting Conjunctions ...

How did Hipparchus (and *"all the arithmeticians"*) calculate Schröder numbers??

> On the Shoulders of Hipparchus:
> A Reappraisal of Ancient Greek Combinatorics.
> — **F. Acerbi** *(2004)*

Why should we be interested?

> *Chrysippus' main achievement is the development of a propositional logic & deductive system[2]. He was innovative in topics central to contemporary formal and philosophical logic. The many close similarities with Gottlob Frege are especially striking.*
> — **Stanford Encyclopedia of Philosophy**

How & why did Chrysippus & Hipparchus come up with such different values?

> Combinatorics for Stoic Conjunction:
> Chrysippus Vindicated, Hipparchus Refuted.
> *Oxford Studies in Ancient Philosophy,* **S. Bobzien** *(2011)*

[2] best understood as a substructural backwards-working Gentzen-style natural-deduction system — **S.E.P.**

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A misunderstanding of logic?

Bobzien's claim is that, "*Hipparchus, it seems, got his mathematics right. What I suggest in this paper is that he got his Stoic logic wrong.*"

Where it starts going wrong :

"He counts the *same sequence* of conjuncts but with *different bracketing* as different conjunctions ... He counts

$$[P \wedge Q] \wedge R \quad \underline{\qquad P \wedge Q \wedge R \qquad} \quad P \wedge [Q \wedge R]$$

as different assertibles. Unlike modern propositional logic, Hipparchus assumes that a [elementary] conjunction can consist of <u>two or more</u> conjuncts.

In order to get to [the little Schröder number 103049], Hipparchus also had to take the order of the ten atomic assertibles as fixed.'" – S. B.

A synthesis via category theory

Between 'equal' and 'not equal' lies a compromise :

$$\text{The Same} \quad \frac{\textit{equal up to}}{\textit{unique isomorphism}} \quad \text{Different}$$

This should be understood at the level of **semantic models**.

What might we need ?

- A family of k-ary **elementary conjunctions**

$$(\_ \star \_) \quad , \quad (\_ \star \_ \star \_) \quad , \quad (\_ \star \_ \star \_ \star \_) \quad , \quad \ldots$$

  (presumably, functors . . .)
- Under substitution / operadic composition these should freely generate an operad isomorphic to $\mathbb{RPT}$.
- A notion of 'equivalence up to natural isomorphism' that <u>uniquely</u> relates any two composites of the same arity.

We need to take into account (lack of) structural rules :

Stoic Sequent Logic and Proof Theory *History & Philosophy of Logic* (2019)

"Much of Stoic logic appears surprisingly modern: a recursively formulated syntax . . . analogues to cut rules, axiom schemata and Gentzen's negation-introduction rules. . . a **deliberate rejection of Thinning [Weakening]**"

What about contraction / idempotency of conjunction?

"Non-simple [assertibles] are those that are, as it were, double ($\delta\iota\pi\lambda\alpha$) – put together by means of a connecting particle from two different assertibles, or **an assertible that is taken twice** ($\delta\iota\zeta$)." — Sextus Empiricus (M8), quoted in S. B.

For convenience, we will :

borrow & generalise a model of conjunction from Linear Logic.

(As a bonus!) Everything is based on **Euclidean division**.

A substructural / relevance logic ! ?

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“Everything is *[in the endomorphism monoid of]* Numbers”

In J.-Y. Girard’s Geometry of Interaction system (Parts 0 – 2) :

Propositions are modelled by functions on $\mathbb{N}$.

- Bijections in the symmetric group $\mathcal{S}(\mathbb{N})$ for MLL
- Partial injections in the symmetric inverse monoid $\mathcal{I}(\mathbb{N})$ for MELL

Conjunction is modelled by the following operation :

$$\begin{aligned}(f \star g)(2n) &= 2.f(n) \\ (f \star g)(2n+1) &= 2.g(n)+1\end{aligned}$$

A simple description, based on Hilbert’s Grand Hotel

This “writes two functions as a single function”, by

replicating their behaviour on the **even** *and* **odd** *numbers respectively.*

This is an injective homomorphism $\mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$, and indeed a categorical (semi-monoidal) tensor

Potential vs. actual infinity

> *The Greeks feared infinity and tried to avoid it ... According to tradition, they were frightened off by the paradoxes of Zeno. ... Until the late C^{19th}, mathematicians were reluctant to accept infinity as more than "potential".*
>
> — *J. Stillwell,* **Mathematics and Its History** *2012*

Not Euclid There exists an infinite number of primes.

Euclid The prime numbers are more numerous than any proposed multitude of prime numbers.

Actual infinity was eventually forced by the requirements of medieval theology:

Duns Scotus on God (R. Cross, 2005)

John Duns Scotus (1266-1308) [ontological] argument may be summarised as,

"If God is composed of parts, then each part must be finite or infinite. ... **If any given part is infinite, then it is equal to the whole**, *which is absurd"*

John Duns' **absurdity** was was (mostly!) stripped of theological interpretations, and taken as a **definition** by G. Cantor.

Not strictly the same . . .

In general : $(P \star Q) \star R \neq P \star (Q \star R)$

No faithful tensor on a non-abelian monoid can be strictly associative.

Coherence & Strictification for Self-Similarity
Journal Homotopy & Related Structures *(P.M.H. 2016)*

There is a non-trivial natural isomorphism $(\_ \star (\_ \star \_)) \Rightarrow ((\_ \star \_) \star \_)$

$$\alpha(a \star (b \star c)) = ((a \star b) \star c)\alpha \quad \forall\, a, b, c \in \mathcal{S}(\mathbb{N})$$

whose unique component (the associator) $\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$

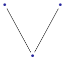
satisfies MacLane's pentagon condition

$$\alpha^2 = (\alpha \star Id)\, \alpha\, (Id \star \alpha)$$

A Hipparchus-style generalisation

Girard gave a **binary** model of conjunction $(\_ \star \_) : \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$.

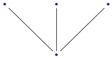
"$(a \star b)$ replicates a, b on the modulo classes $2\mathbb{N}$, $2\mathbb{N}+1$ respectively".

- We draw this as 

There is an obvious **ternary** analogue, $(\_ \star \_ \star \_) : \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \times \mathcal{S}(\mathbb{N}) \hookrightarrow \mathcal{S}(\mathbb{N})$

$$(a \star b \star c)(3n+i) = \begin{cases} 3.a(n) & i=0 \\ 3.b(n)+1 & i=1 \\ 3.c(n)+2 & i=2 \end{cases}$$

"Replicate a, b, c on the modulo classes $3\mathbb{N}$, $3\mathbb{N}+1$, $3\mathbb{N}+2$ respectively".

- We draw this as 

The general case :

For any $k \geqslant 1$, we form the k^{th} **elementary conjunction** by :

$$(f_0 \star \ldots f_{k-1})(kn+i) \;=\; k.f_i(n)+i \;\text{ where } i=0,1,2,\ldots,k-1$$

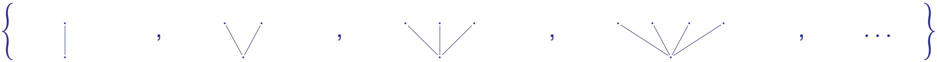
Alternatively & equivalently,

$$(f_0 \star \ldots f_{k-1})(x) \;=\; k.f_i\left(\frac{x-i}{k}\right)+i \;\text{ where } x \equiv i \mod k$$

This gives, for any $k>0$, an injective group homomorphism $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$ that :

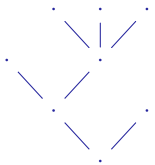
> *"replicates the action of f_0 , f_1 , ... , f_{k-1} on the modulo classes $k\mathbb{N}$, $k\mathbb{N}+1$, ... , $k\mathbb{N}+(k-1)$ respectively.*

For $k=1,2,3,4,\ldots$, we draw these as



Composing elementary conjunctions

These 'compose by substitution' to give an operad $\mathcal{H}ipp$ of **generalised conjunctions**. Each k-leaf tree determines an injective hom. $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$.

 $: \ \mathcal{S}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{S}(\mathbb{N})$

$$(f_0, f_1, f_2, f_3, f_4) \quad \mapsto \quad ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$$

More formally :

We have one operation of each arity > 0 in the (non-symmetric) endomorphism operad of $\mathcal{S}(\mathbb{N})$ within the category $(\mathbf{Grp}, \times)$ of groups / homomorphisms with Cartesian product.

These generate the sub-operad $\mathcal{H}ipp$.

Claim :

The operad $\mathcal{H}ipp$ of generalised conjunctions extends Girard's operation from the Geometry of Interaction, to provide a semantic model for Hipparchus' (mis-)understanding of Chrysippus' Stoic Logic.

More concisely(!)

$\mathcal{H}ipp \cong \mathbb{RPT}$, so each tree determines a *distinct* homomorphism $\mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$.

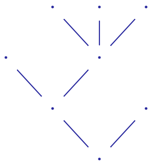
Proof?

Proving this requires a concept the Greeks (notoriously) did not have[3]:

> *The greatest calamity in the history of science was the failure of Archimedes to invent positional notation.* – C. F. Gauss

[3] . . . but may (occasionally) have borrowed from their neighbours

Let me see you counting like they do in Babylon



defines a homomorphism : $\mathcal{S}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{S}(\mathbb{N})$

In the operadic composite $(f_0, f_1, f_2, f_3, f_4) \mapsto ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$, the action of each f_j is mapped :

from The whole of the natural numbers $\mathbb{N}$

to Some modulo class $A_j\mathbb{N} + B_j$.

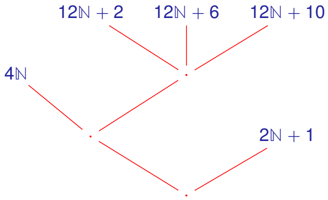
For example : f_3 is translated onto $12\mathbb{N} + 10$.

Question:

How do we derive these coefficients *from the tree*?

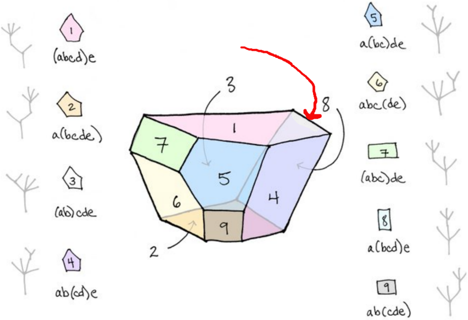
Leaves as modulo classes

label the leaves of a chosen edge
of $\mathcal{K}_5$ by modulo classes



The Fifth Associahedron $\mathcal{K}_5$

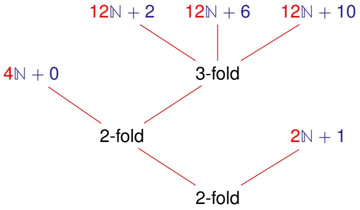
(Diagram "borrowed" from Tai-Danae Bradley's `www.math3ma.com` *blog.)*



All modulo classes are <u>disjoint</u>. Their <u>union</u> is the whole of $\mathbb{N}$.

The multiplicative coefficients

Multiplicative coefficients



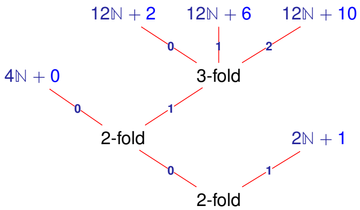
In leaf-traversal ordering

$$\begin{aligned} 4 &= 2 \times 2 \\ 12 &= 2 \times 2 \times 3 \\ 12 &= 2 \times 2 \times 3 \\ 12 &= 2 \times 2 \times 3 \\ 2 &= 2 \text{ (!)} \end{aligned}$$

To find multiplicative parts . . .

Multiply the arities of each branching, from root to leaf.

Additive coefficients



Path from leaf to root

$0 =$

Base 2	Base 2
0	0

$2 =$

Base 3	Base 2	Base 2
0	1	0

$6 =$

Base 3	Base 2	Base 2
1	1	0

$10 =$

Base 3	Base 2	Base 2
2	1	0

$1 =$

Base 2
1

To find additive parts . . .

Write down the 'address' of each leaf[a]
& treat it as a number in a mixed-radix counting system
(with bases determined by the number of branchings).

[a] in **leaf-to-root** order!

A Root & Branch approach

Rooted Planar Trees are uniquely determined by the "addresses" of their leaves, which uniquely determine (ordered) **exact covering systems**.

Heavily studied by P. Erdös (1950s)

Sets of pairwise-disjoint modulo classes, whose union is the whole of $\mathbb{N}$

This is based on **mixed-radix counting systems**

First formal study by G. Cantor, *Über einfache Zahlensysteme* (1869)

(Corol: **Distinct trees determine distinct homomorphisms**).

Mappings between between gen. conjunctions

Given generalised conjunctions $T, U : \mathcal{S}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{S}(\mathbb{N})$, can we find a (well-behaved) natural isomorphism between them?

$$\begin{tikzcd} \mathcal{S}(\mathbb{N})^{\times k} \arrow[rr, bend left, "T"] \arrow[rr, bend right, "U"'] \arrow[rr, phantom, "\Downarrow ??"] & & \mathcal{S}(\mathbb{N}) \end{tikzcd}$$

A simplification : As generalised conjunctions are *monoid* homomorphisms, natural transformations have a *single* component.

We identify nat. iso.s with their unique components in $\mathcal{S}(\mathbb{N})$.

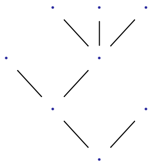
A fuller analysis of natural transformations in the single object setting :

"Monoidal Categories — a unifying concept in math., physics, & C.S."
Noson Yanofsky (M.I.T. Press *appearing shortly*)

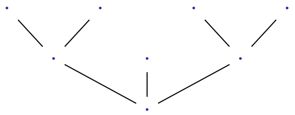
Congruential functions as natural isomorphisms

Consider the generalised conjunctions[4] $T, U : \mathcal{S}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{S}(\mathbb{N})$

$T =$



$U =$



We build a natural isomorphism $\eta_{T,U} : T \Rightarrow U$ by monotonically mapping between their respective ordered covering systems :

leaf 0	$4\mathbb{N}$	$\mapsto$	$6\mathbb{N}$
leaf 1	$12\mathbb{N}+2$	$\mapsto$	$6\mathbb{N}+3$
leaf 2	$12\mathbb{N}+6$	$\mapsto$	$3\mathbb{N}+1$
leaf 3	$12\mathbb{N}+10$	$\mapsto$	$6\mathbb{N}+2$
leaf 4	$2\mathbb{N}+1$	$\mapsto$	$6\mathbb{N}+4$

This gives, as desired,

$$\eta_{T,U}.((a \star (b \star c \star d)) \star e) \;=\; ((a \star b) \star c \star (d \star e)).\eta_{T,U}$$

[4] edges of the fifth associahedron $\mathcal{K}_5$

Observe that :

- $\eta_{T,T} = Id \in \mathcal{S}(\mathbb{N})$
- $\eta_{T,U}\eta_{S,T} = \eta_{S,U}$
- $\eta^{-1}_{T,U} = \eta_{U,T}$

We have a **posetal groupoid** $\mathcal{C}hrys$ of functors / natural iso.s, given by :

Objects Generalised conjunctions (operations of $\mathcal{H}ipp$)

$$\text{Arrows} \quad \mathcal{C}hrys(T,U) = \begin{cases} \{\eta_{T,U}\} & T, U \text{ have the same arity,} \\ \varnothing & \text{otherwise.} \end{cases}$$

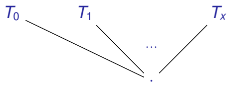
As this groupoid is posetal, *all diagrams commute*.

Unbiased tensors on a posetal groupoid

We may equip $\mathcal{C}hrys$ with a family of **unbiased tensors**, one of each arity :

Given homomorphisms $T_0, \ldots, T_x \in Ob(\mathcal{C}hrys)$, we define

$$(T_0 \otimes \ldots \otimes T_x) \overset{def.}{=}$$



Rather neatly (but entirely expectedly) :

The unique arrow $T_0 \otimes \ldots \otimes T_x \Rightarrow U_0 \otimes \ldots \otimes U_x$ is given by

$$\eta_{(T_0 \otimes \ldots \otimes T_x),(U_0 \otimes \ldots \otimes U_x)} = \left(\eta_{T_0,U_0} \star \ldots \star \eta_{T_x,U_x}\right)$$

Generalised conjunction defines $\mathbb{N}^+$-indexed family of functors on a posetal groupoid :

$$\left\{ (\_ \otimes \ldots \otimes \_) : \prod_k Chrys \rightarrow \mathcal{C}hrys \right\}_{k \in \mathbb{N}^+}$$

Very special case: $\mathcal{C}hrys$ contains a copy of (a unitless version of) MacLane's posetal monoidal groupoid $(\mathcal{W}, \_\square\_)$.

Concrete formulæ for arrows of $\mathcal{C}hrys$

Given two ordered exact covering systems, determined by k-ary generalised conjunctions T, U

$$\begin{array}{cccc} \textbf{leaf } 0 & A_0\mathbb{N} + B_0 & \mapsto & C_0\mathbb{N} + D_0 \\ \textbf{leaf } 1 & A_1\mathbb{N} + B_1 & \mapsto & C_1\mathbb{N} + D_1 \\ \vdots & \vdots & & \vdots \\ \textbf{leaf } k-1 & A_{k-1}\mathbb{N} + B_{k-1} & \mapsto & C_{k-1}\mathbb{N} + D_{k-1} \end{array}$$

The natural isomorphism $\eta_{T,U}$ is the bijection

$$\eta_{T,U}(x) = \frac{1}{A_j}\left(C_j x + \begin{vmatrix} A_j & B_j \\ C_j & D_j \end{vmatrix} \right) \text{ where } x \equiv B_j \mod A_j$$

We arrive at **congruential functions**, introduced in

"Unpredictable Iterations" — J. Conway (1972)

to demonstrate undecidability in elementary arithmetic.

Bobzien's 'three simple assertibles' example

$$\begin{tikzcd}[column sep=small]
((_ \star _) \star _) & & (_ \star (_ \star _)) \arrow[ll, Rightarrow, "\alpha"'] \\
& (_ \star _ \star _) \arrow[ul, Rightarrow, "\gamma_\flat"] \arrow[ur, Rightarrow, "\gamma"'] &
\end{tikzcd}$$

$$\gamma_\flat(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \mod 3, \\ \frac{4n+2}{3} & n \equiv 1 \mod 3, \\ \frac{2n-1}{3} & n \equiv 2 \mod 3. \end{cases} \qquad \gamma(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3. \end{cases}$$

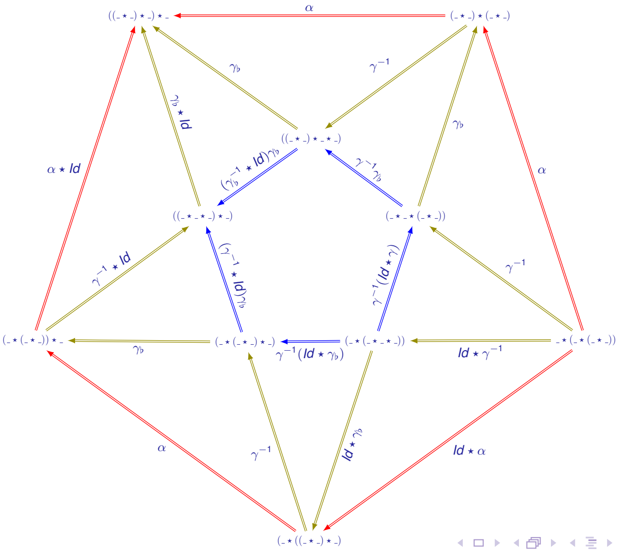
$$\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$$

These are familiar from *other areas* :

- α : the **associator** for the conjunction from G.O.I.
- γ : the **amusical permutation** from a (unresolved) conjecture of Collatz.
- $\gamma_\flat$: the **flattened permutation**, defined by $1 + \gamma_\flat(n) = \gamma(n+1)$.

From the third to the fourth associahedron

We can label (most of) $\mathcal{K}_4$ using conjunctions / composites of labels from $\mathcal{K}_3$.



Context for the "amusical permutation"

Conway's classic *Unpredictable Iterations* (1972) paper[5] exhibited

undecidability of iterative problems on **congruential functions**.

He discussed his motivation in (among other places) :

Unsettleable Arithmetic Problems – J. Conway 2012

"What is the simplest Collatz-style game that we can expect to be undecidable? I think I have an answer!" — J. C.

[5] See also Sergei Maslov, *On E. L. Post's Tag Problem* (1964)

What is this "simplest undecidable game"?

The $3x + 1$ problem & its generalisations – Jeffrey Lagarias (1985)

Writing about L. Collatz : *"In his notebook dated July 1, 1932, he considered the function*

$$\gamma(n) = \begin{cases} \frac{2}{3}n & \textit{if } n \equiv 0 \ (\textit{mod } 3) \\ \frac{4}{3}n - \frac{1}{3} & \textit{if } n \equiv 1 \ (\textit{mod } 3) \\ \frac{4}{3}n + \frac{1}{3} & \textit{if } n \equiv 2 \ (\textit{mod } 3) \end{cases}$$

He posed the problem of whether the cycle containing 8 *is finite or infinite. I will call this the* **Original Collatz Conjecture**. *His original question has never been answered.*

John Conway called this bijection the **Amusical Permutation** & claimed the OCC as

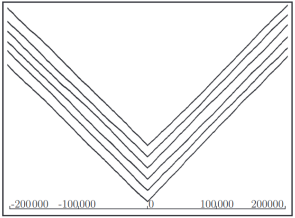
"The simplest undecidable (& therefore 'true') arithmetic statement."

Melodic Conjectures?

A “probvious” conjecture!

> *“The proportion of fallacies in published proofs is far greater than the small positive probability that [this conjecture is false]”*
>
> – *J.C.,* Unsettleable Algebraic Problems *(2012)*

A plot of $n \; : \; log\left(\gamma^n(k)\right)$, for $k = 8, 14, 40, 64, 80, 82$



$$\gamma^{200000}(8) \approx 10^{5000}$$

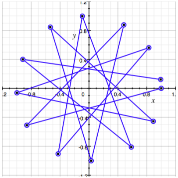
It goes like this, the fourth, the fifth ...

> *"There are twelve notes per octave, which represents a doubling of frequency, just as twelve steps [of γ or γ^{-1}] approximately doubles a number, on average."* — J.C. (2012)

This average-case doubling is not **exact** :

- [The amusical permutation] doubles by a factor of $\frac{3^{12}}{2^{18}} \approx 2$
- [Its inverse] doubles by a factor of $\frac{2^{20}}{3^{12}} \approx 2$

"A frequency ratio of $\frac{3^{12}}{2^{19}}$ is called the **Pythagorean comma** and is the difference between enharmonically equivalent notes (e.g. $A^\sharp$ and $B^\flat$). So there really is a connection with music."



Exact doubling / the octave is given by their geometric mean $\sqrt{\frac{3^{12}}{2^{18}} \cdot \frac{2^{20}}{3^{12}}} = 2$.

The amusing musical permutation

"Since the series always ascends by a fifth, modulo octaves, it does not sound very musical. It has always amused me to call it amusical."