Towards comonadic locality theorems

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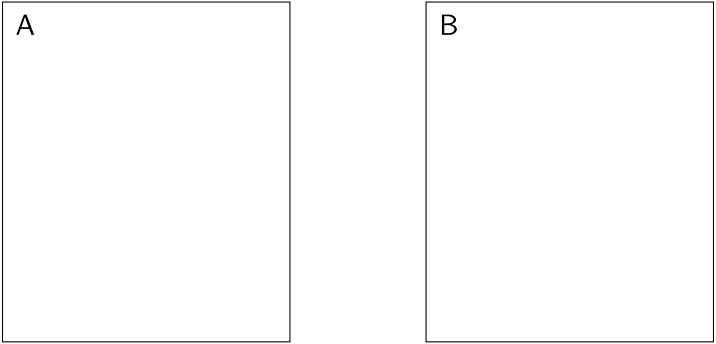
18 September 2023

Workshop on Springer Volume "Samson Abramsky on Logic and Structure in Computer Science and Beyond", London

k-round Ehrenfeucht–Fraïssé game



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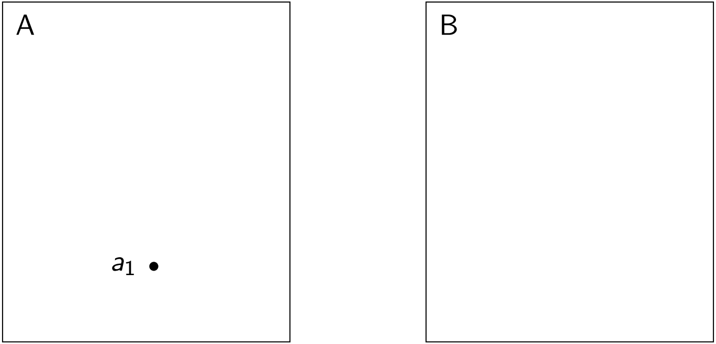

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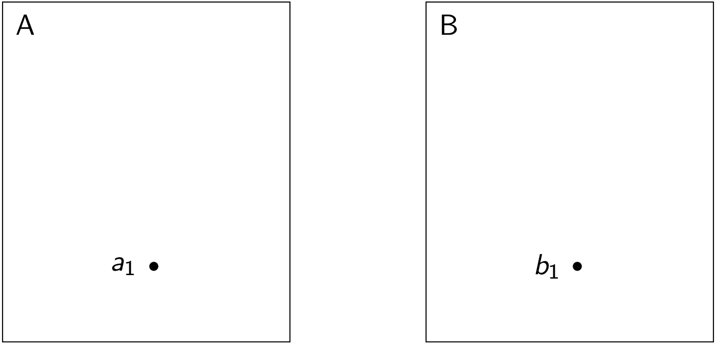

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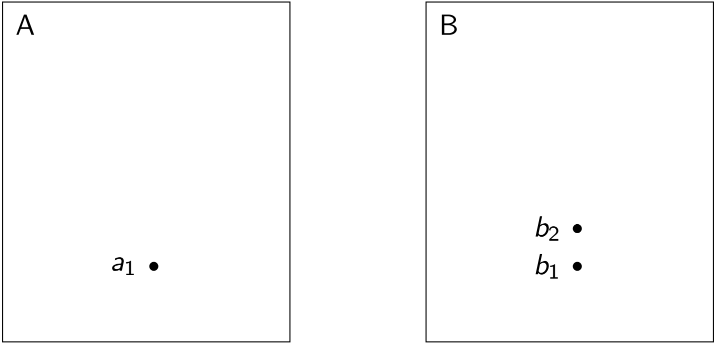

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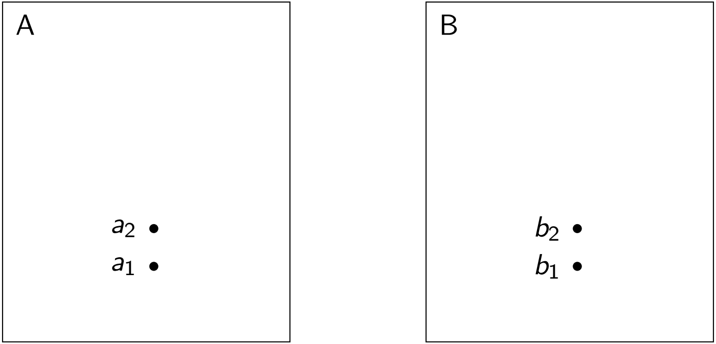

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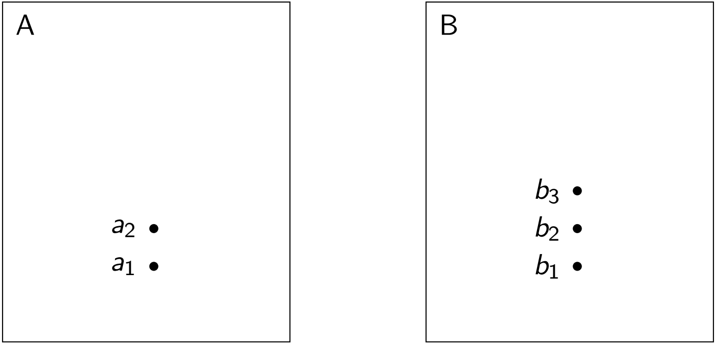

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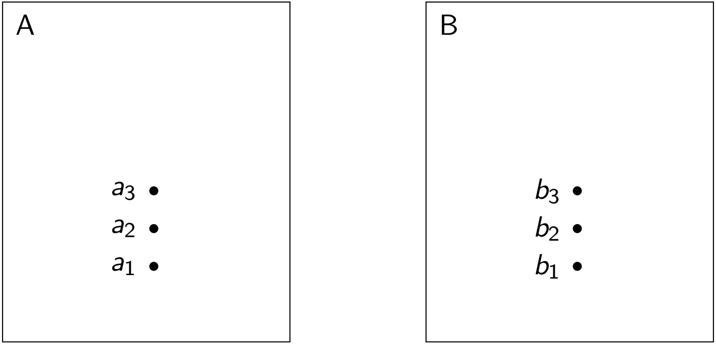

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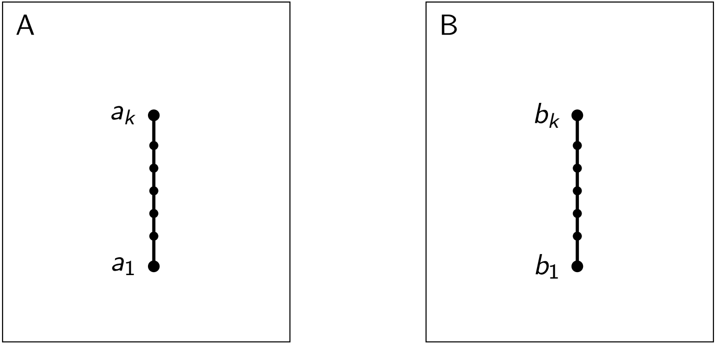

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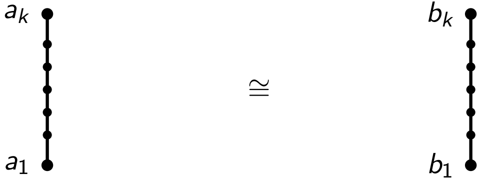

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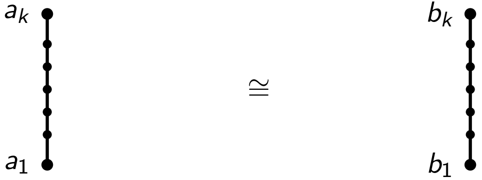

Theorem

$A \equiv_k B$ iff Duplicator wins in the k-round E–F game.

Game comonads in a nutshell

For a (well-behaved) model comparison game for logic $\mathscr{L}$

- exploration of one structure following the rules of the game
  $\Rightarrow$ construction $\mathbb{C}\colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma)$

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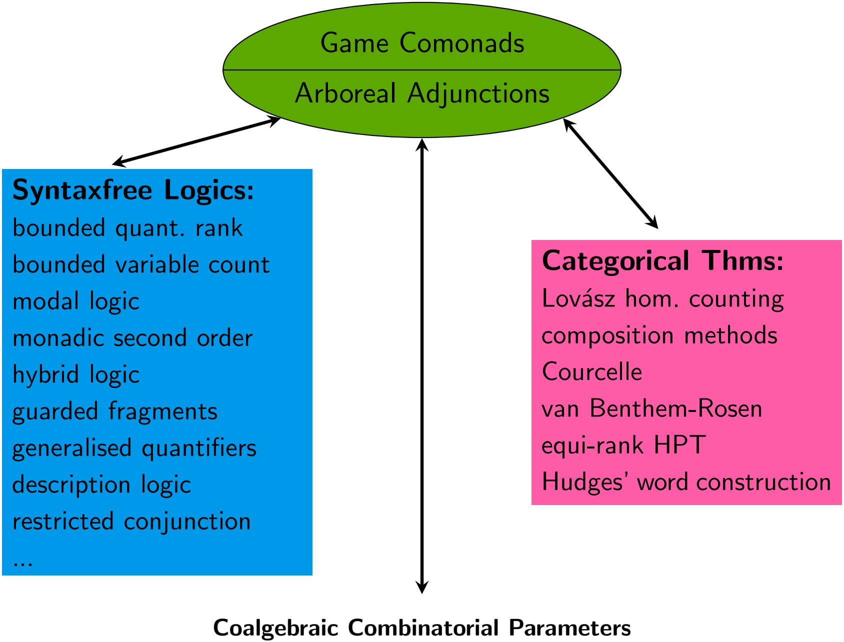
- free coalgebra $F(A) \approx (\mathbb{C}(A), \sqsubseteq)$
- a **bisimulation** $F(A) \sim F(B)$ iff $A \equiv_{\mathscr{L}} B$

- bisimulation expressed in terms of paths and embeddings
- existential (positive), counting fragments also captured in $\mathrm{CoAlg}(\mathbb{C})$

Example: Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$

Given $A \in \mathcal{R}(\sigma)$,

- $\mathbb{E}_k(A) =$ sequences $\overline{a} = [a_1, \ldots, a_n]$ with $a_i \in A$ and $n \leq q$
- $\overline{a} \sqsubseteq \overline{b}$ iff $\overline{a}$ is a prefix of $\overline{b}$
- $\mathrm{CoAlg}(\mathbb{E}_k) \approx \sigma$-structures with a compatible forest order
- $A \equiv_k B$ iff $F(A) \sim F(B)$ (for $U \dashv F$ arising from $\mathbb{E}_k$)


Game Comonads
Arboreal Adjunctions
Syntaxfree Logics:
bounded quant. rank
bounded variable count
modal logic
monadic second order
hybrid logic
guarded fragments
generalised quantifiers
description logic
restricted conjunction
...
Categorical Thms:
Lovász hom. counting
composition methods
Courcelle
van Benthem-Rosen
equi-rank HPT
Hudges' word construction
Coalgebraic Combinatorial Parameters

Categorical Skeleton + Combinatorial Core = Theorems

Lovász homomorphism counting theorems:

- comonadicity + preservation of finiteness/finite rank

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- provides many new examples!
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Question: Significant missing theorems? ... locality theorems!

Locality theorems

Omnipresent in finite model theory. We need them too!

Theorem (Gaifman, 1982)

For relational structures: $A \equiv^{\mathrm{local}}_{r(k),q(k)} B$ *implies* $A \equiv_k B$.

$A \equiv^{\mathrm{local}}_{r,q} B$ is equivalence under **basic local sequences**

$$\exists x_1, \ldots, x_n \, (\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \theta(x_i))$$

of qrank $\leq q$ where θ is r-**local**:

$$A \models \theta(a) \quad \text{iff} \quad \mathcal{N}_r(a) \models \theta(a).$$

$\mathcal{N}_r(x) = \{y \mid \delta(x, y) \leq r\}$

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Theorem (Hanf, 1965)

For graphs A and B with finite neighbourhoods, bijection of isomorphism $\mathcal{N}_r$-types up to ω implies $A \equiv B$.

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Theorem (Fagin–Stockmeyer–Vardi, 1995)

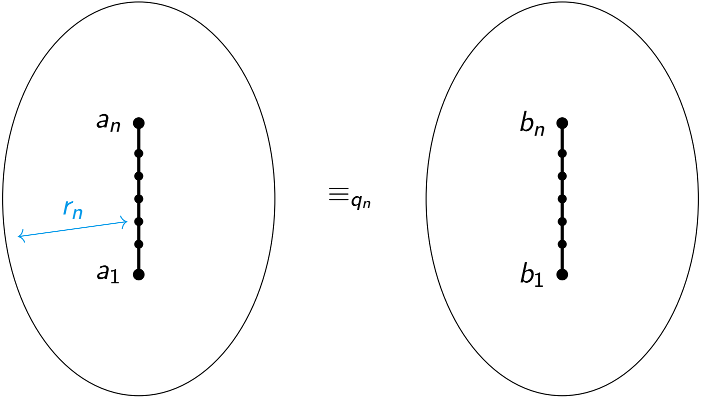
For finite A and B with neighbourhoods $\leq f$, *bijection of isomorphism* $\mathcal{N}_{r(k)}$*-types up to* $w(f,k)$ *implies* $A \equiv_k B$.

Proof structure

We fix suitable radii $r_1, \dots, r_k$ and quantifier ranks $q_1, \dots, q_k$.

(typically $q_i \approx c^{k-i}$)

Invariant at round n**:**



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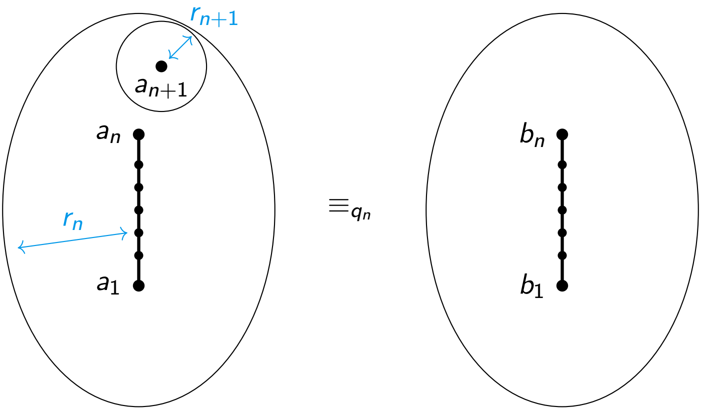

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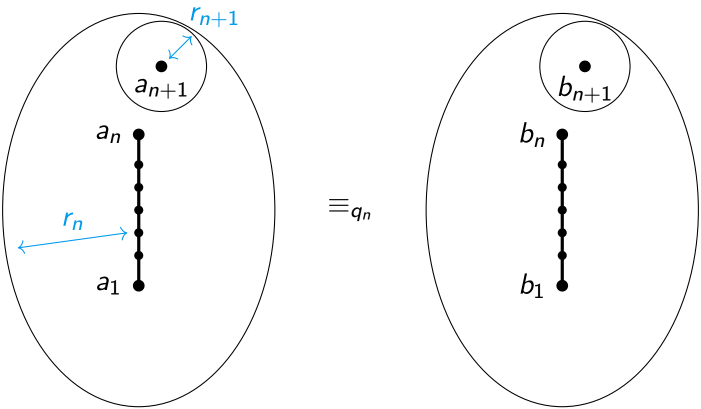

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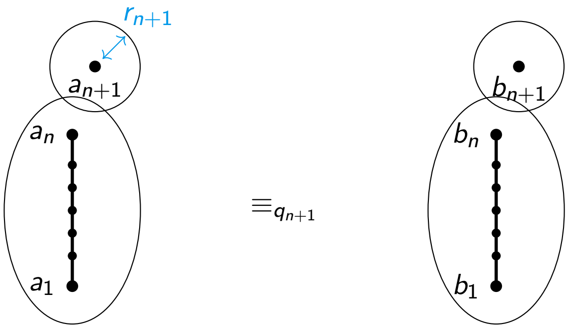

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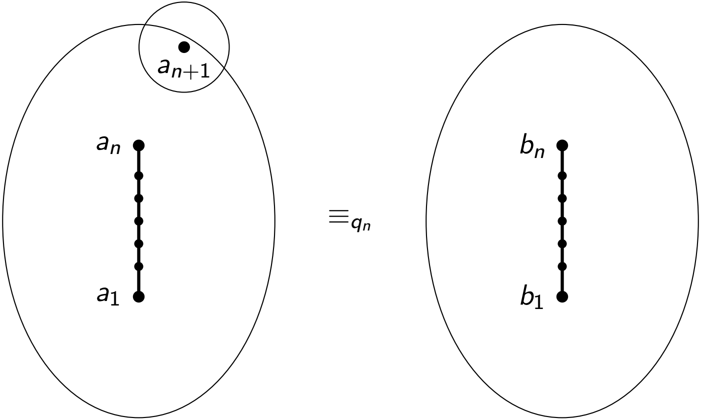

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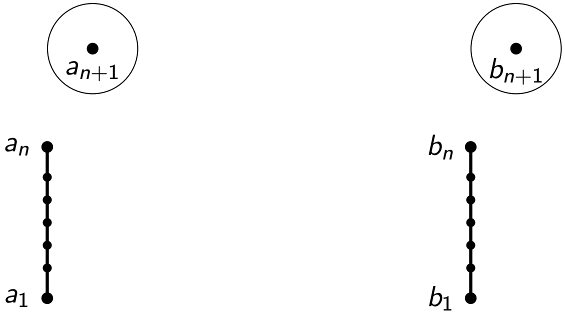

Productivity step: use $\equiv^{\text{local}}_{r_n,q_n}$ to find b_{n+1}

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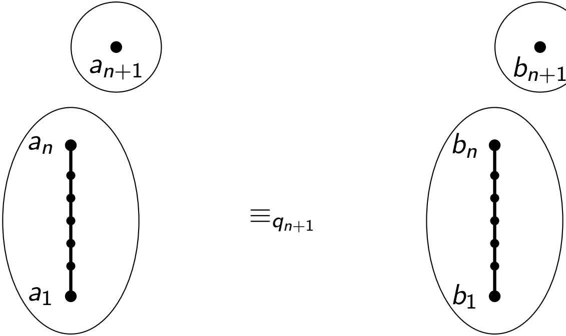

Comonadic proof structure

$$\mathsf{CoAlg}(\mathbb{C}) \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{R}(\sigma)$$

Want:

$$F(A) \sim F(B) \text{ in } \mathsf{CoAlg}(\mathbb{C})$$

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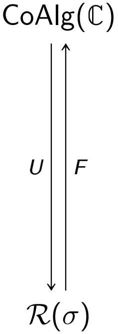

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Instead of $r_1, \ldots, r_k$ and $q_1, \ldots, q_k$ we have $\mathcal{N}_1, \ldots, \mathcal{N}_k$ and $\mathbb{D}_1, \ldots, \mathbb{D}_k$.

abstract neighbourhood operators, e.g. $\mathcal{N}(x) = \{y \mid \delta(x, y) \leq r\}$

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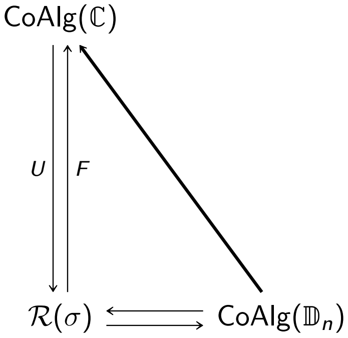

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Invariant at round n:

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in CoAlg($\mathbb{D}_n$)

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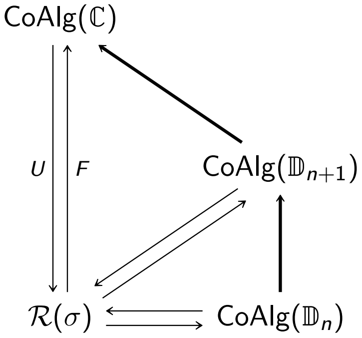

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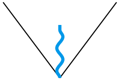
$\overline{a} \in A \quad \approx$  $\approx \quad P \xrightarrow{\overline{a}} F(A)$ for P "discrete"

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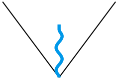
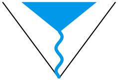
$\overline{a} \in A \quad \approx$  $\approx \quad P \xrightarrow{\overline{a}} F(A)$ for P "discrete"

Model theoretic types: $\quad \mathtt{tp}(\overline{a}) \quad \approx$  $\subseteq F(A)$

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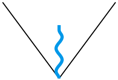
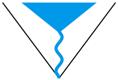
$\overline{a} \in A \quad \approx$  $\approx \quad P \xrightarrow{\overline{a}} F(A)$ for P "discrete"

Model theoretic types: $\quad \mathtt{tp}(\overline{a}) \quad \approx$  $\subseteq F(A)$

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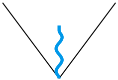
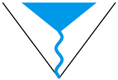
$\overline{a} \in A \quad \approx$  $\approx \quad P \xrightarrow{\overline{a}} F(A)$ for P "discrete"

Model theoretic types: $\quad \mathtt{tp}(\overline{a}) \quad \approx$  $\subseteq F(A)$

Lemma. $(A, \overline{a}) \equiv (B, \overline{b})$ iff $\mathtt{tp}(\overline{a}) \sim \mathtt{tp}(\overline{b})$

Neighbourhood operator: $\overline{a} \in A \mapsto \mathcal{N}(\overline{a}) \subseteq A$

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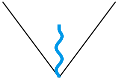
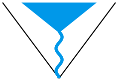
$\overline{a} \in A \quad \approx$  $\approx \quad P \xrightarrow{\overline{a}} F(A)$ for P "discrete"

Model theoretic types: $\quad \mathtt{tp}(\overline{a}) \quad \approx$  $\subseteq F(A)$

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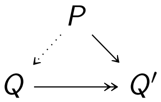
Lemma. $(\mathcal{N}(\overline{a}), \overline{a}) \equiv (\mathcal{N}(\overline{b}), \overline{b}) \quad$ iff $\quad \mathtt{ltp}_{\mathcal{N}}(\overline{a}) \sim \mathtt{ltp}_{\mathcal{N}}(\overline{b})$

Axioms

A path P is **discrete** iff it is projective wrt quotients:



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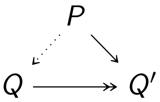

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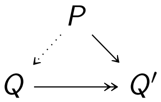

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Combinatorics of neighbourhoods

Given $\mathcal{N}(\overline{a}) \subseteq A$ and $\mathcal{N}(\overline{b}) \subseteq A$, the factorisation

$$\mathcal{N}(\overline{a}) + \mathcal{N}(\overline{b}) \twoheadrightarrow \mathcal{N}(\overline{a} \mathbin{+\!\!+} \overline{b}) \rightarrowtail A$$

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Combinatorics of neighbourhoods

Given $\mathcal{N}(\overline{a}) \subseteq A$ and $\mathcal{N}(\overline{b}) \subseteq A$, the factorisation

$$\mathcal{N}(\overline{a}) + \mathcal{N}(\overline{b}) \twoheadrightarrow \mathcal{N}(\overline{a} \mathbin{+\!\!+} \overline{b}) \rightarrowtail A$$

Lemma. For $\mathcal{N}(x) = \mathcal{N}_r(x)$, $\delta(\overline{a}, \overline{b}) > 2 \cdot r$ iff $\twoheadrightarrow$ above is iso.

Theorem (Categorical Skeleton)

Given a comonad $\mathbb{C}$, opmonoidal comonads $\mathbb{D}_1, \mathbb{D}_2, \ldots$ and neighbourhood operator $\mathcal{N}_1, \mathcal{N}_2, \ldots$ such that

- *there are strong comonad morphisms $\mathbb{C} \Rightarrow \mathbb{D}_{i+1} \Rightarrow \mathbb{D}_i$*
- CoAlg($\mathbb{C}$) *has enough splitting discrete paths*

If, for every i, A, B satisfies the Productivity Condition from $\mathbb{D}_i, \mathcal{N}_i$ to $\mathbb{D}_{i+1}, \mathcal{N}_{i+1}$ then $A \equiv_{\mathbb{C}} B$.

The Productivity Condition intuitively:

$$\mathtt{ltp}_{\mathcal{N}_i}(\overline{a}) \sim \mathtt{ltp}_{\mathcal{N}_i}(\overline{b}) \text{ and } \mathcal{N}_{i+1}(a_{i+1}) \not\subseteq \mathcal{N}_i(\overline{a})$$

$$\Longrightarrow \exists b_{i+1} \text{ s.t. } \quad \text{“ } \delta(b_{i+1}, \overline{b}) > 2 \cdot \mathcal{N}_{i+1} \text{ ”}$$

$$\mathtt{ltp}_{\mathcal{N}_{i+1}}(a_{i+1}) \sim \mathtt{ltp}_{\mathcal{N}_{i+1}}(b_{i+1})$$

Final words

Productivity Condition easy to check for the Workspace Lemma!

A categorical proof of Gaifman and Hanf also possible.
Although, it requires further axioms for discrete paths and neighbourhood operators.

Next steps:

- fully axiomatic van Benthem-Rosen
- algorithmic results which use locality
- nowhere-dense comonads in terms of locality assumptions