# CSP Relaxations and the Hierarchy of Contextuality

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# Why this topic?

# Circa 2019

# An innocent looking open question

## The Quantum Monad on Relational Structures

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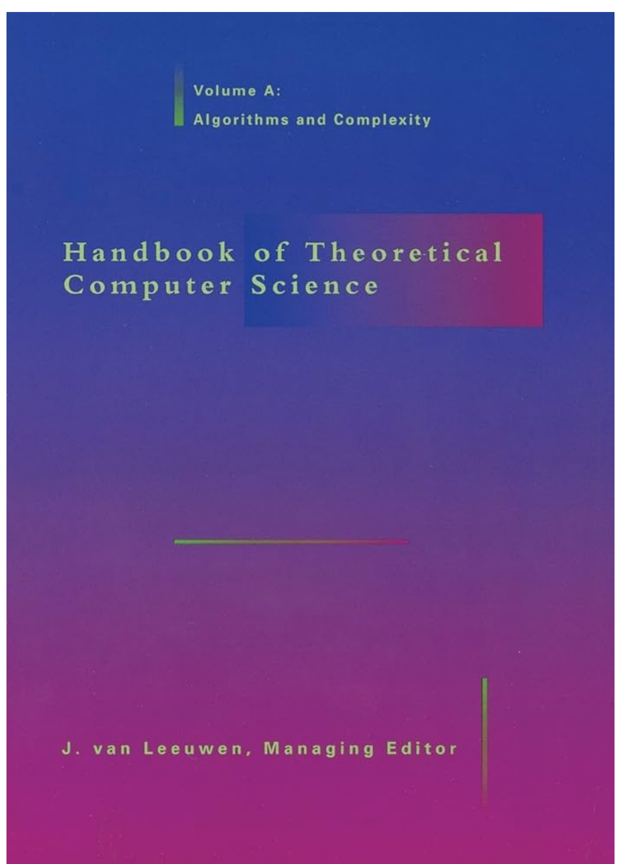
- Can other concepts from finite model theory, such as pebble games, which admit a comonadic formulation [7], be similarly quantized?

# Track A vs Track B

- We spent a lot of our first meeting discussing the divide in theoretical computer science research. It goes by many names, “Track A vs Track B”, “Power vs Structure”, “US theory vs Euro theory”, etc.

- At some point I learnt from Samson that the divide can be traced back to the “Handbook of Theoretical Computer Science” published in the 1980s.




# Example: $MIP^* = RE$

- Non-local games have been studied extensively by physicists since at least the pioneering work of Bell in the 1960s.
- Interactive proof systems have been studied by computer scientists at least since 1980s.
- Yet, to the best of my knowledge the first time the observation was explicitly made that these abstract frameworks are mathematically equivalent came in 2004!

# Example: $MIP^* = RE$

- After a lot of work this observation, which brought to disparate fields together, led to the remarkable result $MIP^* = RE$. Which is simultaneously a big result in three different fields:

  1. Complexity Theory: Allowing provers to share entangled resources gives them dramatically more computational power.

  2. Quantum Foundations: Provides a solution to Tsirelson's problem, which asks if the set of correlations produced in the commuting operator framework of quantum mechanics is equivalent to those produced by the tensor product framework.

  3. Operator Algebras: Solves Conne's Embedding Conjecture.

# Bringing Together Disparate Fields

We have witnessed up close, on enjoyable afternoons spent exchanging in Samson's office, or over the occasional cheeky pint, his impressive breadth of knowledge, his ability to see further and to spot or establish connections between disparate fields, even across traditional disciplinary boundaries. We have been led to recognise those boundaries not as ontological but as imagined and thus re-imaginable. And in his company we have experienced an environment in which common language can be found and shared among people of very different backgrounds. Testament to this capacity for bringing people together is the present set of authors, including by original academic background a computer scientist, a mathematician, and a physicist.

[1]: Barbosa, Rui Soares, Martti Karvonen, and Shane Mansfield. "Closing Bell: Boxing black box simulations in the resource theory of contextuality." *arXiv preprint arXiv:2104.11241* (2021).

# Motivating Question

- We can associate to any empirical model $e$ a CSP $K_e$. Then we have:

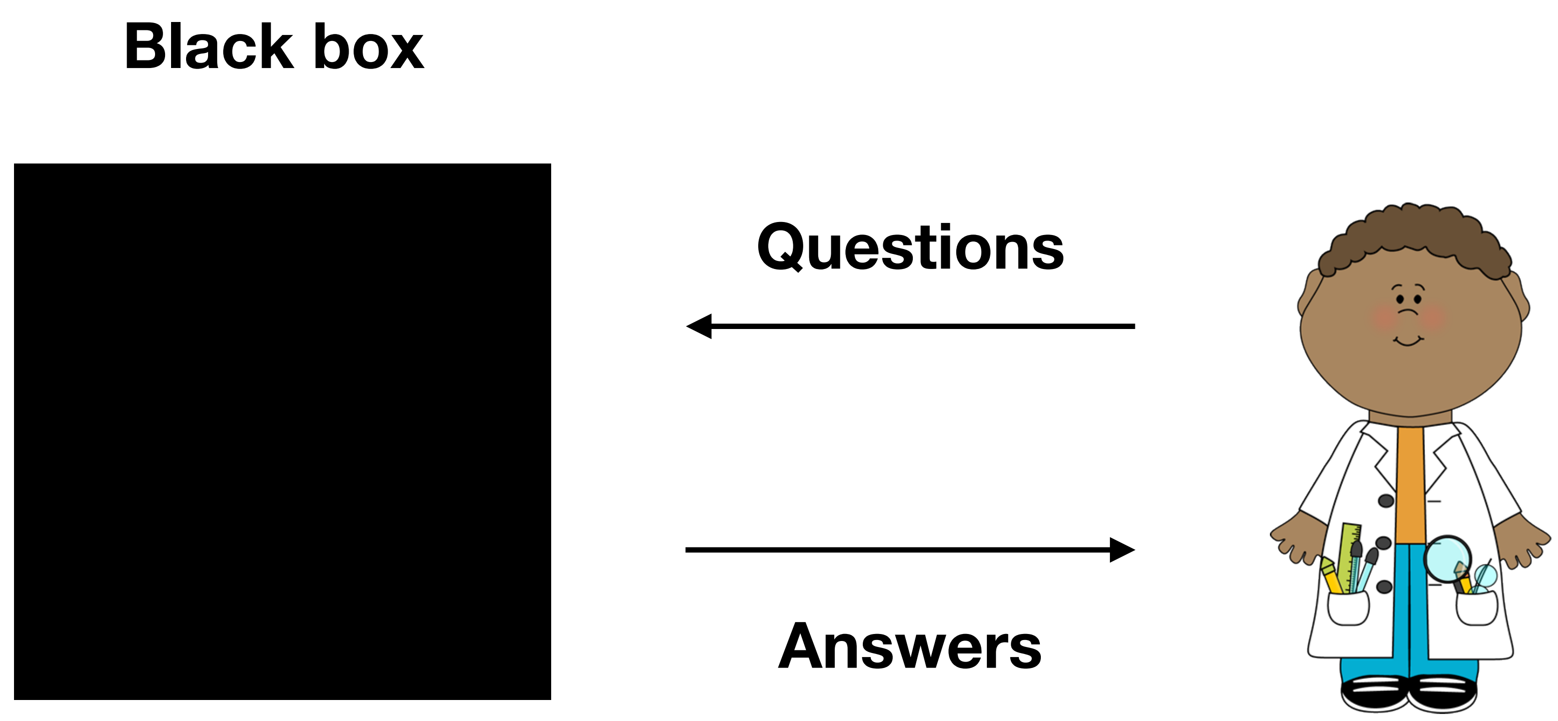
**Proposition 13.** *There is a one-to-one correspondence between consistent global assignments for $e$ and solutions of $\mathcal{K}_e$. Thus $e$ is strongly contextual iff $\mathcal{K}_e$ has no (classical) solution.*

**Do other levels of the hierarchy of contextuality have analogues in the CSP world?**


[1]: Abramsky, Samson, and Adam Brandenburger. "The sheaf-theoretic structure of non-locality and contextuality." *New Journal of Physics* 13.11 (2011): 113036.

[2] Abramsky, Samson, et al. "The quantum monad on relational structures." *arXiv preprint arXiv:1705.07310* (2017).

# The Experimental Setup



# Formalisation [1]

| A | B | (0, 0) | (1, 0) | (0, 1) | (1, 1) |
|---|---|---|---|---|---|
| $a$ | $b$ | 1/2 | 0 | 0 | 1/2 |
| $a'$ | $b$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a$ | $b'$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a'$ | $b'$ | 1/8 | 3/8 | 3/8 | 1/8 |

[1]: Abramsky, Samson, and Adam Brandenburger. "The sheaf-theoretic structure of non-locality and contextuality." *New Journal of Physics* 13.11 (2011): 113036.

# Formalisation [1]

Measurement Scenario

Outcome

| A | B | (0, 0) | (1, 0) | (0, 1) | (1, 1) |
|---|---|---|---|---|---|
| $a$ | $b$ | 1/2 | 0 | 0 | 1/2 |
| $a'$ | $b$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a$ | $b'$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a'$ | $b'$ | 1/8 | 3/8 | 3/8 | 1/8 |

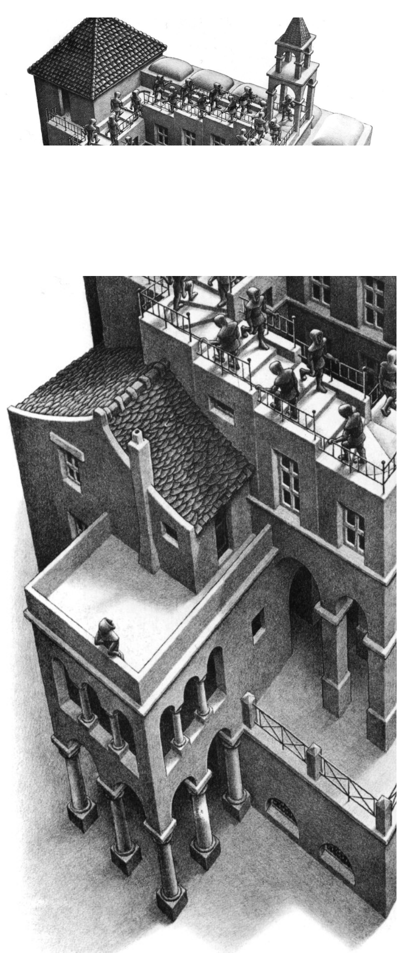
Empirical Model

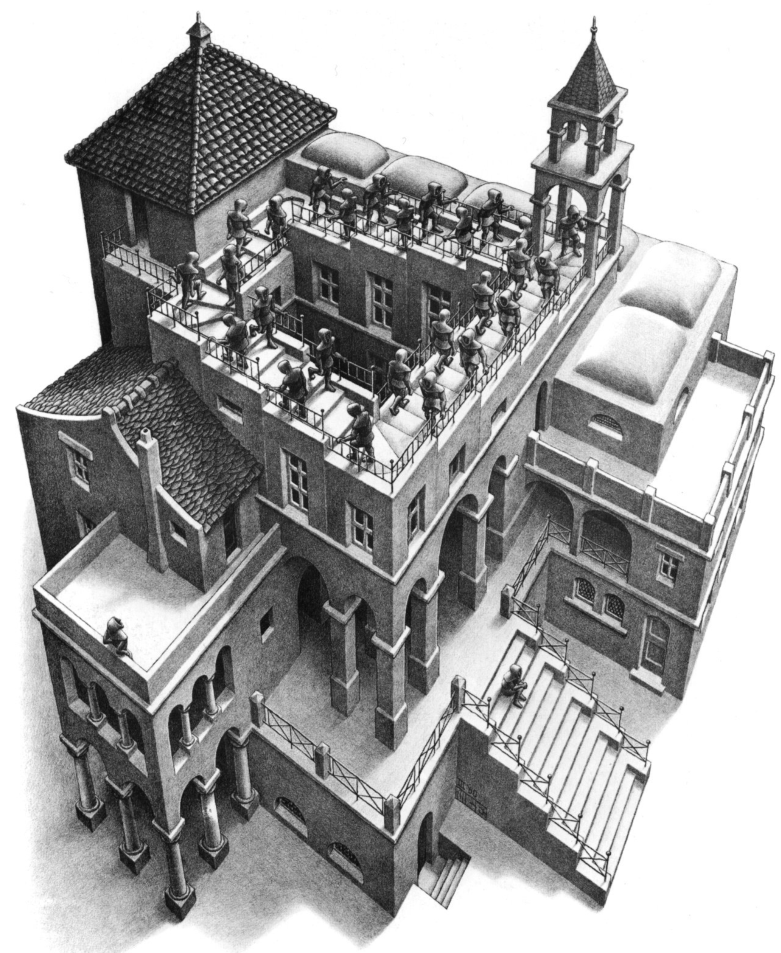
Measurement

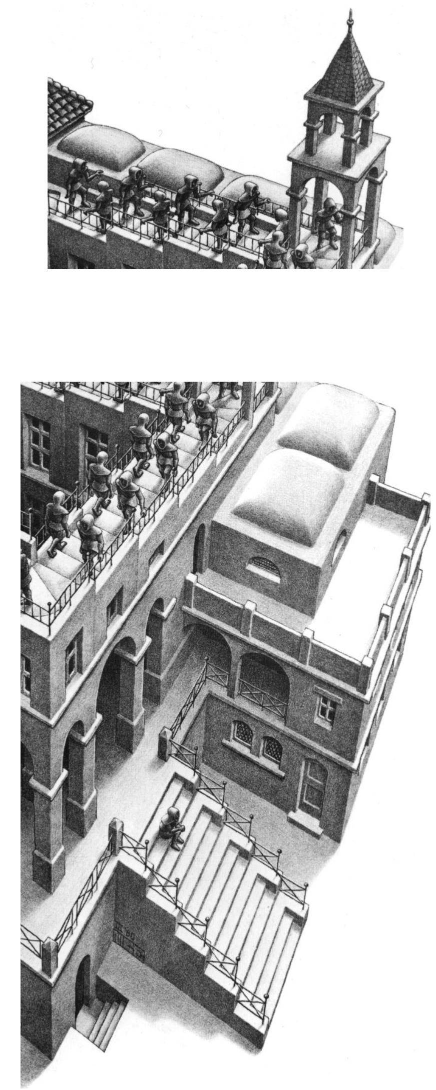
[1]: Abramsky, Samson, and Adam Brandenburger. "The sheaf-theoretic structure of non-locality and contextuality." *New Journal of Physics* 13.11 (2011): 113036.

# Contextuality

**Slogan:** Local consistency but global inconsistency.







[1]: Cervantes, Víctor H., and Ehtibar N. Dzhafarov. "Contextuality analysis of impossible figures." *Entropy* 22.9 (2020): 981.

# Contextuality

| A | B | (0,0) | (1,0) | (0,1) | (1,1) |
|---|---|---|---|---|---|
| $a$ | $b$ | 1/2 | 0 | 0 | 1/2 |
| $a'$ | $b$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a$ | $b'$ | 3/8 | 1/8 | 1/8 | 3/8 |
| $a'$ | $b'$ | 1/8 | 3/8 | 3/8 | 1/8 |

- A model is ***contextual*** if no distribution over global assignment of outcomes to measurements restricts via marginalisation to the data in the table.

# Formalisation [1]

| $a, b, a', b'$ | Prob. |
|---|---|
| 0000 | $1/8$ |
| 0011 | $1/4$ |
| 0100 | $1/4$ |
| 1000 | $1/4$ |
| 1010 | $1/8$ |
| Other | 0 |

| A | B | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $1/2$ | 0 | 0 | $1/2$ |
| $a'$ | $b$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
| $a$ | $b'$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
| $a'$ | $b'$ | $1/8$ | $3/8$ | $3/8$ | $1/8$ |

- For example, the assignment above disagrees with the first row of the model since $P((a,b) \to (0,0)) = 1/8 + 1/4 = 3/8 \neq 1/2$

Screenshot taken from Caru, G. Logical and Topological Contextuality in Quantum Mechanics and Beyond. University of Oxford, 2019.

# Formalisation [1]

| $a, b, a', b'$ | Prob. |
|---|---|
| 0000 | $1/8$ |
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| 1000 | $1/4$ |
| 1010 | $1/8$ |
| Other | 0 |

| A | B | $(0,0)$ | $(1,0)$ | $(0,1)$ | $(1,1)$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $1/2$ | 0 | 0 | $1/2$ |
| $a'$ | $b$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
| $a$ | $b'$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
| $a'$ | $b'$ | $1/8$ | $3/8$ | $3/8$ | $1/8$ |

- In fact, no global assignment agrees with this table. Thus, this empirical model is contextual.

# Logical Contextuality

| A | B | 00 | 01 | 10 | 11 |
|---|---|---|---|---|---|
| $x_1$ | $y_1$ | 1 | 1 | 1 | 1 |
| $x_1$ | $y_2$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_1$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_2$ | 1 | 1 | 1 | 0 |

- A model is ***logically contextual*** if no boolean distribution over global assignment of outcomes to measurements agrees with its possibilistic collapse.

- The example above is known as Hardy's Paradox [1, 2].

[1]: Hardy, Lucien. "Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories." *Physical Review Letters* 68.20 (1992): 2981.

[2]: Hardy, Lucien. "Nonlocality for two particles without inequalities for almost all entangled states." *Physical Review Letters* 71.11 (1993): 1665.

# Strong Contextuality

| A | B | 00 | 01 | 10 | 11 |
|---|---|---|---|---|---|
| $x_1$ | $y_1$ | 1 | 1 | 1 | 1 |
| $x_1$ | $y_2$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_1$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_2$ | 1 | 1 | 1 | 0 |

- A model is ***strongly contextual*** if there is not even a single global assignment of boolean values which is consistent with its possibilistic collapse.

- Hardy's model is not strongly contextual. Consider $\{x_1 \to 1, y_1 \to 1, x_2 \to 0, y_2 \to 0\}$.

# Strong Contextuality

| A | B | 00 | 01 | 10 | 11 |
|---|---|---|---|---|---|
| $x_1$ | $y_1$ | $1/2$ | 0 | 0 | $1/2$ |
| $x_1$ | $y_2$ | $1/2$ | 0 | 0 | $1/2$ |
| $x_2$ | $y_1$ | $1/2$ | 0 | 0 | $1/2$ |
| $x_2$ | $y_2$ | 0 | $1/2$ | $1/2$ | 0 |

- The example above is known as a PR box [1]. It is strongly contextual but not quantum realisable.

[1]: Popescu, Sandu, and Daniel Rohrlich. "Quantum nonlocality as an axiom." *Foundations of Physics* 24 (1994): 379-385.

# Hierarchy of Contextuality

Bell < Hardy < PR box

Probabilistic < Logical < Strong

# Constraint Satisfaction Problem (CSP)

- A **CSP instance** is a triple $\langle X, D, C \rangle$ where:

  1. $X = \{X_1, \ldots, X_n\}$ is a set of variables.

  2. $D$ is a set of domains of values for the variables.

  3. $C = \{C_1, \ldots C_m\}$ is a set of constraints. Each constraint is itself a pair $\langle T, R \rangle$ where $T \subseteq X$ is a k element subset of variables and $R \subseteq D^k$.

- A **solution** to a CSP is a function $f : X \to D$ such that for all $\langle t, R \rangle \in C$ $f(t) \in R$.

# CSP example

- $X = \{x_1, x_2, y_1, y_2\}$, $D = \{0,1\}$, $C = \{C_1, C_2, C_3, C_4\}$. Where:

1. $C_1 = x_1, y_1 -> \{00,11\}$
2. $C_2 = x_1, y_2 -> \{00,11\}$
3. $C_3 = x_2, y_1 -> \{00,11\}$
4. $C_1 = x_2, y_2 -> \{01,10\}$

- This particular CSP has no solution. It corresponds to the PR box example.

# Connection with contextuality

Domain Element

Variable

| A | B | 00 | 01 | 10 | 11 |
|---|---|---|---|---|---|
| $x_1$ | $y_1$ | 1 | 1 | 1 | 1 |
| $x_1$ | $y_2$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_1$ | 0 | 1 | 1 | 1 |
| $x_2$ | $y_2$ | 1 | 1 | 1 | 0 |

Constraint

- Possiblistic empirical models correspond to CSP instances.
- A model is strongly contextual iff the corresponding CSP has no solution.

# The Homomorphism Problem

- Here is a relatively well-known fact:

**Proposition 12.** *There is a one-to-one correspondence between homomorphisms* $\mathcal{A} \to \mathcal{B}$ *and solutions for* $\mathcal{K}_{AB}$.

- So instead of thinking about CSPs directly we can think of homomorphisms of relational structures.

# System of Equations

- Solving the following system of equations is equivalent to deciding if a homomorphism $A \to B$ exists.

$$
\begin{aligned}
\forall a \in A, b \in B \; x_{a,b} \in \{0,1\} & & & \\
\mathbf{a} \in R^{\mathcal{A}}, \mathbf{b} \in B^{ar(R)} \; x^R_{\mathbf{a},\mathbf{b}} \in \{0,1\} & & & \\
\forall a \in A & & \sum_{b \in B} x_{a,b} = 1 & \qquad \text{(Hom.1)} \\
\forall \mathbf{a} \in R^{\mathcal{A}}, \mathbf{b} \notin R^{\mathcal{B}} & & x^R_{\mathbf{a},\mathbf{b}} = 0 & \qquad \text{(Hom.2)} \\
\forall \mathbf{a} \in R^{\mathcal{A}}, a \in \{\mathbf{a}\}, b \in B & & \sum_{\substack{f : \{\mathbf{a}\} \to B \\ f(a) = b}} x^R_{\mathbf{a}, f(\mathbf{a})} = x_{a,b} & \qquad \text{(Hom.3)}
\end{aligned}
$$

This particular presentation is due to Adam Ó Conghaile

# System of Equations

- We can relax these equations by using the structure of a semiring.

$$
\begin{array}{rlr}
\forall a \in A, b \in B \ x_{a,b} \in \mathbf{S} & & \\
\mathbf{a} \in R^{\mathcal{A}}, \mathbf{b} \in B^{ar(R)} \ x^{R}_{\mathbf{a},\mathbf{b}} \in \mathbf{S} & & \\
\forall a \in A & \displaystyle\sum_{b \in B} x_{a,b} = 1_{\mathbf{S}} & (LP_{\mathbf{S}}.1) \\
\forall \mathbf{a} \in R^{\mathcal{A}}, \mathbf{b} \notin R^{\mathcal{B}} & x^{R}_{\mathbf{a},\mathbf{b}} = 0_{\mathbf{S}} & (LP_{\mathbf{S}}.2) \\
\forall \mathbf{a} \in R^{\mathcal{A}}, a \in \{\mathbf{a}\}, b \in B & \displaystyle\sum_{\substack{g\colon \{\mathbf{a}\} \to B \\ g(a) = b}} x^{R}_{\mathbf{a},g(\mathbf{a})} = x_{a,b} & (LP_{\mathbf{S}}.3)
\end{array}
$$

# Distribution Monads

- We saw in Nihil's talk that in the category of sets a version of the distribution monad $D$ can be defined for any semiring.

- It turns out that the relaxations we are interested in can be modelled using versions of $D$ defined on the category of relational structures $R(\sigma)$.

**Definition 6.** *The distribution monad over $R(\sigma)$ is defined as follows:*

- *The universe of $\mathbb{D}(\mathcal{A})$ is the set of all functions $\phi\colon A \to [0,1]$ where $\sum_{a \in A} \phi(a) = 1$.*
- $R^{\mathbb{D}}_{\mathcal{A}} = \{(\sum_{\boldsymbol{a} \in R^{\mathcal{A}}} \gamma_{\boldsymbol{a}} \boldsymbol{a}[1] \ldots, \sum_{\boldsymbol{a} \in R^{\mathcal{A}}} \gamma_{\boldsymbol{a}} \boldsymbol{a}[m]) \mid \gamma : R^{\mathcal{A}} \to [0,1], \sum_{\boldsymbol{a} \in R^{\mathcal{A}}} \gamma_{\mathcal{A}} = 1\}$
- $\eta_X(x) = 1.x.$
- $\mu_{\mathcal{A}}(\psi)(a) = \sum_{\phi \in \mathbb{D}A} \psi(\phi).\phi(a).$

This particular presentation is due to Adam Ó Conghaile

# BLP and Arc Consistency

- We will be particularly interested in relaxations given by the following semirings:

    1. The positive real numbers
    2. The Boolean semiring

This particular presentation is due to Adam Ó Conghaile

# BLP and Arc Consistency

**Theorem:** The following are equivalent

1. $A \to \mathbb{D}B$.
2. $BLP(A, B)$ has a solution.

**Theorem:** The following are equivalent

1. $A \to \mathbb{P}B$.
2. $AC(A, B)$ has a solution.

This particular presentation is due to Adam Ó Conghaile

# Aside: Connections With Comonads?

The arc consistency relaxation coincides with the 2 pebble game. That is to say:

$$A \to \mathbb{P}B \iff P_2 A \to B$$

**Is this a fluke or a manifestation of some deeper connection between monadic and comonadic relaxations of the homomorphism problem?**

# Back to Contextuality

**Proposition 13.** *There is a one-to-one correspondence between consistent global assignments for e and solutions of $\mathcal{K}_e$. Thus e is strongly contextual iff $\mathcal{K}_e$ has no (classical) solution.*

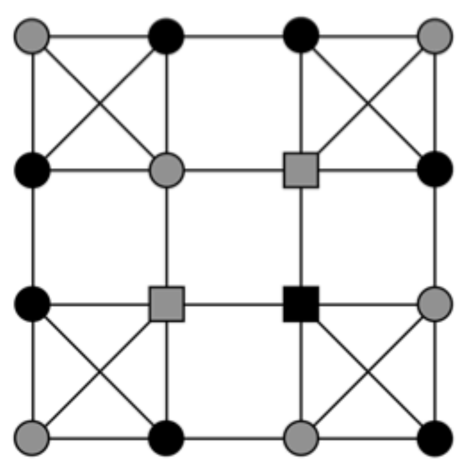
- We can now state analogous results for probabilistic and logical contextuality.

**Proposition:** There is a one-to-one correspondence between global sections for $e$ and BLP solutions for $\mathscr{K}_e$. Thus e is contextual iff $\mathscr{K}_e$ has no BLP solution.

**Proposition:** There is a one-to-one correspondence between boolean global sections for $e$ and BLP solutions for $\mathscr{K}_e$. Thus e is logically contextual iff $\mathscr{K}_e$ has no AC solution.

# Graph-Theoretic Hierarchy of Contextuality

- We can associate to any measurement scenario an “exclusivity graph” [1, 2].
- (Behind the scenes this exclusivity graph really arises as the primal graph of a hypergraph whose hyperedges are contexts)



| | | | |
|---|---|---|---|
| 00 | 10 | 01 | 11 |
| $A_0B_0$ | | $A_1B_0$ | |
| 01 | 11 | 00 | 10 |
| 10 | 00 | 11 | 01 |
| $A_0B_1$ | | $A_1B_1$ | |
| 11 | 01 | 10 | 00 |


[1]: Cabello, Adán, Simone Severini, and Andreas Winter. "Graph-theoretic approach to quantum correlations." *Physical review letters* 112.4 (2014): 040401.

[2]: de Silva, Nadish. "Graph-theoretic strengths of contextuality." *Physical Review A* 95.3 (2017): 032108.

# Summary

| Sheaf Framework | CSP | Graph Invariant | |
|---|---|---|---|
| Probabilistic Contextuality | No BLP solution | Weighted Independence Number | |
| Logical Contextuality | No arc-consistency solution | Minimal Independence Number | |
| Strong Contextuality | No solution | Independence Number | |
| | | | |

# Future Work

- Various forms of reductions between CSPs have been extensively studied in computer science

- It is fairly clear that there is a correspondence between the existence of such CSP reductions and the existence of simulations between different empirical models.

- Are there deeper consequences to this correspondence? For instance, does the existence of efficient CSP reductions say anything interesting from a physics point of view?