Ripples from Pebbles

Dan Marsden

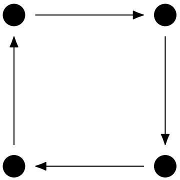
September 2023

Aims

- Introduce the pebbling comonad.
- Briefly discuss joint work with Samson on guarded and bounded fragments.
- Report on very recent ongoing developments from this years adjoint school.

Pebble Games

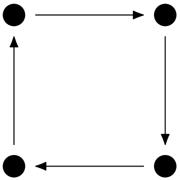
Four Cycle



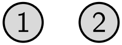
Three Path

Pebble Games

Four Cycle



Spoiler Pebbles



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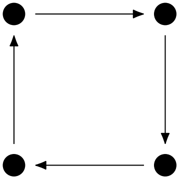

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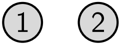
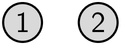


Duplicator Pebbles



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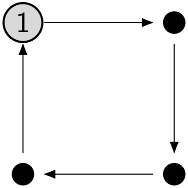

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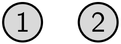

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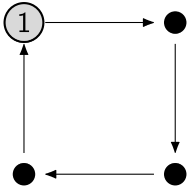

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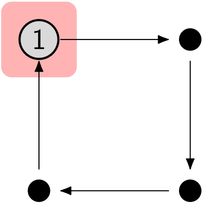

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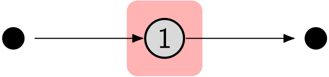

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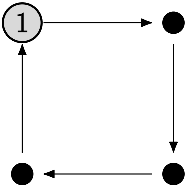

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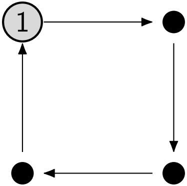

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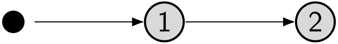

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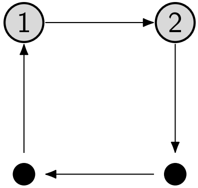

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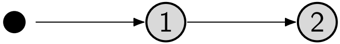

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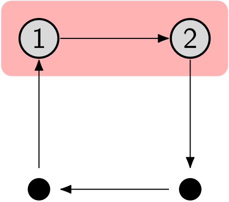

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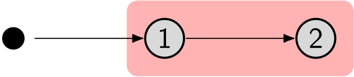

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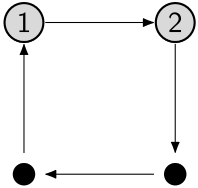

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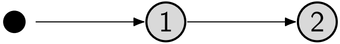

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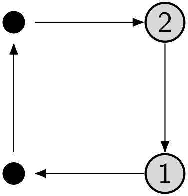

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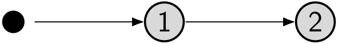

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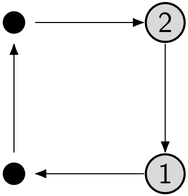

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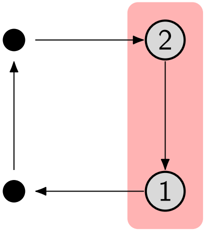

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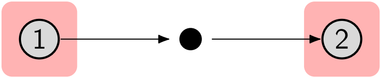


Spoiler wins!

Games and Logic

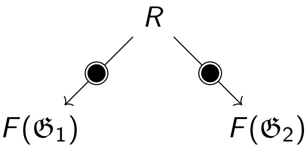
For graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$:

- Duplicator has a winning strategy in the k-pebble game if and only if
$$\mathfrak{G}_1 \equiv_{\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2.$$
- Duplicator has a winning strategy in both k-pebble games in which Spoiler is constrained to play in a fixed graph if and only if
$$\mathfrak{G}_1 \equiv_{\exists^+\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2.$$
- Duplicator has a winning strategy in another variant of the k-pebble game if and only if
$$\mathfrak{G}_1 \equiv_{\#\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2.$$

Comonads, Games and Logic

There is a comonad[1] $\mathbb{P}_k : \mathrm{DGraph} \to \mathrm{DGraph}$ such that for directed graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$:

- $\mathfrak{G}_1 \equiv_{\exists^+\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2$ if and only if there is a pair of morphisms in $\mathrm{DGraph}_{\mathbb{P}_k}$
$$\mathfrak{G}_1 \to \mathfrak{G}_2 \quad \text{and} \quad \mathfrak{G}_1 \leftarrow \mathfrak{G}_2.$$
- $\mathfrak{G}_1 \equiv_{\#\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2$ if and only if there is an isomorphism in $\mathrm{DGraph}_{\mathbb{P}_k}$
$$\mathfrak{G}_1 \cong \mathfrak{G}_2.$$
- $\mathfrak{G}_1 \equiv_{\mathcal{L}^k_{\infty,\omega}} \mathfrak{G}_2$ if and only if there exists a suitable span in $\mathrm{DGraph}^{\mathbb{P}_k}$



[1] Abramsky, Dawar, and Wang, "The pebbling comonad in Finite Model Theory".

Sketch of $\mathbb{P}_k$

A graph of plays

For a graph $\mathfrak{G}$:

- We form a new graph $\mathbb{P}_k(\mathfrak{G})$, with vertices non-empty sequences of the form

$$[(p_1, v_1), (p_2, v_2), \ldots, (p_n, v_n)],$$

with each v_i a vertex of $\mathfrak{G}$, and each $1 \leq p_i \leq k$.

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  with each v_i a vertex of $\mathfrak{G}$, and each $1 \leq p_i \leq k$.

- Intuitively two such sequences are part of the same play of the game if one is a prefix of another. For two such sequences, there is an edge

$$[\ldots, (p_m, v_m)] \to [\ldots, (p_n, v_n)]]$$

  if there is an edge $v_m \to v_n$ in $\mathfrak{G}$, and the pebbles p_m and p_n haven't moved again in that play.

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This all generalises from graphs to general relational structures.

Other Logics

A Recurring Pattern[2]

In an analogous way, building structures of plays:

- There is a comonad $\mathbb{E}_k$ that characterises equivalence in bounded quantifier depth first-order logic.
- There is a comonad $\mathbb{M}_k$ that characterises equivalence in the modal fragment.

In both cases, the characterisation includes existential positive and counting games as well.

[2]Abramsky and Shah, "Relating structure and power: Comonadic semantics for computational resources".

Guarded Quantification

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We consider first-order logic, but restricting to quantification of the form:

$$\exists \overline{y}.\ \gamma(\overline{x}, \overline{y}) \wedge \varphi(\overline{x}, \overline{y})$$

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The formula γ is a **guard**. There are three possible guard types:

- **Atom** γ is an atom in which all the variables $\overline{x}$ and $\overline{y}$ appear.
- **Loose** γ is a conjunction of atoms encoding that the variables in $\overline{x}$ and $\overline{y}$ form a clique in the Gaifman graph.
- **Clique** γ is the formula $\mathrm{clique}(\overline{x}, \overline{y})$ stating the variables in $\overline{x}$ and $\overline{y}$ form a clique in the Gaifman graph.

Guarded Quantification

The Guarded Quantification Game

For relational structures $\mathfrak{A}$ and $\mathfrak{B}$:

Round 0 : We set $X_0 := \varnothing$, $Y_0 := \varnothing$, $\varphi_0 := \varnothing$.

Round $n+1$: Spoiler now has two options.

1. Spoiler specifies a guarded set X_{n+1} in $\mathfrak{A}$. Duplicator must respond with a guarded set Y_{n+1} in $\mathfrak{B}$, and a partial isomorphism $\varphi_{n+1} : X_{n+1} \to Y_{n+1}$, such that $\varphi_{n+1}|_X = \varphi_n|_X$, where $X = X_{n+1} \cap X_n$.
2. Spoiler specifies a guarded set Y_{n+1} in $\mathfrak{B}$. Duplicator must respond with a guarded set X_{n+1} in $\mathfrak{A}$, and a partial isomorphism $\varphi_{n+1} : X_{n+1} \to Y_{n+1}$, such that $\varphi_{n+1}^{-1}|_Y = \varphi_n^{-1}|_Y$, where $Y = Y_{n+1} \cap Y_n$.

Winning condition: Baked into the move structure.

Guarded Quantification

Categorical Semantics for Guarded Fragments

For each choice of guard type, we presented[3] a comonad $\mathbb{G}_k$ such that:

- Equivalence in the existential positive guarded logic is characterised by homomorphisms in both directions between structures in the Kleisli category.
- Equivalence in guarded logic is characterised by suitable spans in the Eilenberg–Moore category.
- The fundamental idea to internalise equivalence as a structure encoding plays of a game remains the same, although significantly technically more elaborate.
- Counting equivalence remains open.

[3] Abramsky and Marsden, "Comonadic semantics for guarded fragments".

Bounded Fragments

(Two-sided) Bounded Quantification

We consider the fragment of first-order logic with quantification restricted to the forms:

$$\exists y.\ E(x, y) \land \varphi$$
$$\exists y.\ E(y, x) \land \varphi$$

This can equivalently be described syntactically as a bidirectional hybrid modal logic with additional "memory" operations.

Bounded Fragments

The Bounded Game

Fix pointed relational structures $(\mathfrak{A}, a_0)$ and $(\mathfrak{B}, b_0)$, with distinguished binary relation symbol E. In round n of the bounded game, there are four possible moves:

- **$\mathfrak{A}$ forward** Spoiler chooses $a_{n+1} \in \mathfrak{A}$ such that there exists $m < n$ with $E^{\mathfrak{A}}(a_m, a_n)$. Duplicator responds with $b_{n+1} \in \mathfrak{B}$ such that there exists $m < n$ with $E^{\mathfrak{B}}(b_m, b_n)$.
- **$\mathfrak{A}$ backward** Spoiler chooses $a_{n+1} \in \mathfrak{A}$ such that there exists $m < n$ with $E^{\mathfrak{A}}(a_n, a_m)$. Duplicator responds with $b_{n+1} \in \mathfrak{B}$ such that there exists $m < n$ with $E^{\mathfrak{B}}(b_n, b_m)$.

and the dual moves where Spoiler plays in $\mathfrak{B}$.

Winning condition: The map $a_i \mapsto b_i$ is a partial isomorphism.

Bounded Fragments

Categorical Semantics for the Bounded Fragment

We presented a comonadic account of two-sided bounded quantification[4], including:

- A comonad characterising the existential positive and two-sided games. This comonad is a blend of the comonads for the bounded quantifier depth and modal fragments.
- A van Benthem–Rosen type result giving a semantic characterisation of this fragment.
- A comonadic account of the one-sided bounded fragment remains open, and requires fundamentally new ideas.

[4] Abramsky and Marsden, "Comonadic semantics for hybrid logic".

The Adjoint School Game Comonads Project[5]

We decided to look at computationally "nice" logics, extending modal logic, from the perspective of game comonads.

[5]Students: Tyler Hanks, Zhixuan Yang, Richie Yeung, Elena Dimitriadis Bermejo and TA: Nihil Shah

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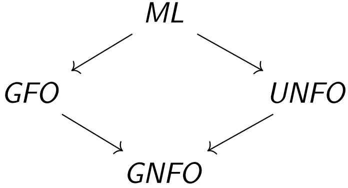

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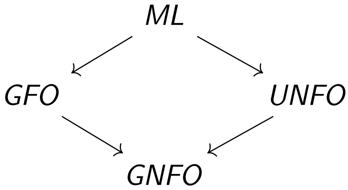

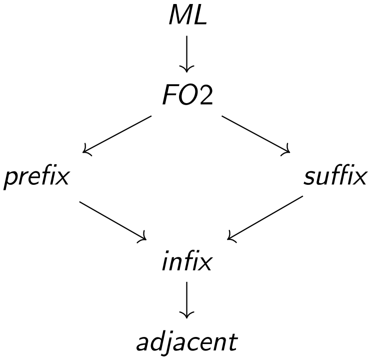

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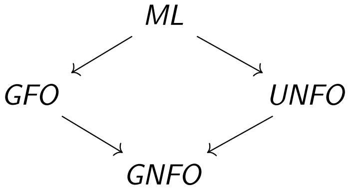

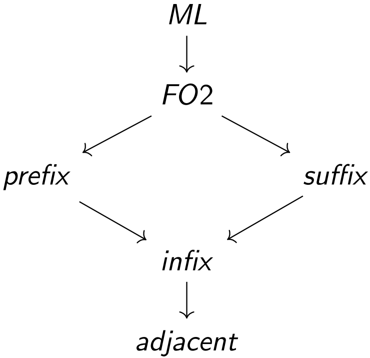

The intention is to get more data points about the nature of game comonads.

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Restricting Negation

We now consider the unary[6] and guarded negation[7] fragments of first-order logic. These have unrestricted $\exists$, and no $\forall$, but with negation only of the form:

$\neg\varphi(x)$	**Unary Negation (UNFO)**
$\gamma(\overline{x}) \wedge \neg\varphi(\overline{x})$	**Guarded Negation (GNFO)**

The game is more complicated than those previously considered in the game comonads programme as it has multiple phases.

[6] Segoufin and ten Cate, "Unary negation".
[7] Bárány, ten Cate, and Segoufin, "Guarded negation".

Guarded and Unary Negation

Ongoing work of the 2023 Adjoint School Game Comonads Project

Initial Steps

To investigate these fragments at the adjoint school, we

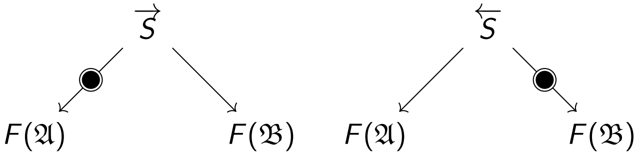
- Restricted attention to UNFO to simplify the problem.
- Established some equivalent games with nicer properties, such as apparent second-order moves, to more closely aligned with more familiar games.

Guarded and Unary Negation

Ongoing work of the 2023 Adjoint School Game Comonads Project

Comonads and UNFO

Our rephrasing of the model comparison game lead us to consider the pebbling comonad $\mathbb{P}_k$, and pairs of spans of Eilenberg–Moore coalgebras the form:



This seems to capture the right notion of two-sided equivalence - ongoing work.

Ordered Variable Fragments

Ongoing work of the 2023 Adjoint School Game Comonads Project

Ordered, Forward and Fluted

We consider fragments of first-order logic without equality, with restricted use of variables in atoms:

$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_2, x_3)$	**fluted** (suffixes)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_2)$	**ordered** (prefixes)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge P(x_2)$	**forward** (subsequences)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_1)$	non example
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There is also a more complex **adjacent fragment**[6] in which sequences can climb and fall.

[6]Bednarczyk, Kojelis, and Pratt-Hartmann, "On the Limits of Decision: the Adjacent Fragment of First-Order Logic".

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$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_2, x_3)$	**fluted** (suffixes)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_2)$	**ordered** (prefixes)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge P(x_2)$	**forward** (subsequences)
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_1)$	non example
$\exists x_1.\ \forall x_2.\ \exists x_3.\ R(x_1, x_2, x_3) \wedge E(x_1, x_3)$	non example

There is also a more complex **adjacent fragment**[6] in which sequences can climb and fall.

[6] Bednarczyk, Kojelis, and Pratt-Hartmann, "On the Limits of Decision: the Adjacent Fragment of First-Order Logic".

Ordered Variable Fragments

Ongoing work of the 2023 Adjoint School Game Comonads Project

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Plan A

We considered a variation of the comonad $\mathbb{E}_k$ for bounded quantifier depth logic, but with restrictions on how the relations are imposed:

- Yields legitimate comonads.
- Seems to be the *right* construction for the fluted fragment.
- Seems to be the *wrong* construction for the other fragments, due to issues with variable rebinding not present in the fluted fragment.

Plan B

By a more careful consideration of the formalisation of the syntax of these fragments, we were lead to a variation of the pebbling comonad $\mathbb{P}_k$ which both:

- Restricts the order the pebbles can be played in.
- Restricts which sequences can have relations between them.

Looks encouraging, yields legitimate comonads, but the connections with logic need further verification.

Further work

Continue establishing data points giving comonadic characterisations of various logics:

- UNFO and GNFO, and potentially the tri-guarded fragment.
- Ordered fragments and the recently introduced adjacent fragment.

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- UNFO and GNFO, and potentially the tri-guarded fragment.
- Ordered fragments and the recently introduced adjacent fragment.
- The uniform one-dimensional fragment.
- It would be nice to get fixed-point logics into the picture.
- Some computationally nastier logics...

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