Relations in Order-enriched Categories

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Introduction

Goal

Do for relations in order-enriched settings what regular categories and allegories (or something like that) do for ordinary relations.

Weakening relation:

A weakening relation is $R \subseteq X \times Y$ s.t.

- if $x \leq x'$, $x' R y'$ and $y' \leq y$, then xRy.

These form a category Rel(Pos):

- Standard relational composition
- Identity on X is $\leq_X$

Other model categories

Meet semilattice relation:

A semilattice relation is a weakening relation between semilattices s.t.

- if xRy and xRy', then $xR(y \wedge y')$;
- $xR\top$.

Frame relation:

A frame relation is a semilattice relation between frames s.t.

- if x_iRy for each $i \in I$, then $\bigvee_{i\in I} x_iRy$.

These, and many similar examples, also form categories Rel(SL) and Rel(Frm).

In each case, the underlying category Pos, etc., is recovered by the left adjoint relations.

Background on ordinary relations

Ordinary regular categories

An ordinary category $\mathcal{A}$ is regular iff

- $\mathcal{A}$ has all finite limits
- $\mathcal{A}$ has strong/mono factorization — will explain later
- strong morphisms are stable under pullbacks

Remarks

Any algebraic quasivariety is a regular category.

Indeed, any regular category that has free objects over sets is a quasivariety.

Categories of relations

In a category with finite limits and strong/mono factorization:

Basic relations

A basic relation is an isomorphism class of jointly monic spans — same as subobjects of $A \times B$.

Any span $A \stackrel{p}{\leftarrow} R \stackrel{q}{\rightarrow} B$ determines a basic relation $[p, q]$ by factoring $\langle p, q \rangle$ — assuming products and factorization.

Relation composition

Composition $\varphi; \psi$ is defined by

- Pick monic spans $A \stackrel{p_R}{\leftarrow} R \stackrel{q_R}{\rightarrow} B$ and $B \stackrel{p_S}{\leftarrow} S \stackrel{q_S}{\rightarrow} C$ representing φ and ψ
- form a pullback (r, s) of (q_R, p_S).
- $\varphi; \psi$ is the relation $[p_R r, q_S s]$.

A Justifying Theorem

Theorem

For an ordinary category $\mathcal{A}$ with finite limits and strong/mono factorization the following are equivalent:

1. Composition (;) is associative.
2. Strong morphisms are stable under pullback — that is, the category is regular

Category of relations

So a regular category $\mathcal{A}$ determines a category of relations $\mathrm{Rel}(\mathcal{A})$.

Actually $\mathrm{Rel}(-)$ extends to a 2-functor provided we define the relevant 2-categories carefully.

Structure of Rel($\mathcal{A}$)

Theorem

For any regular category $\mathcal{A}$, Rel($\mathcal{A}$) is a tabular cartesian bicategory in which all objects satsify the Frobenius axiom.

Moreover, if $\mathcal{R}$ is a tabular cartesian bicategory in which all objects satisfy Frobenius, then there is a regular category Map($\mathcal{R}$), for which $\mathcal{R} \equiv \mathrm{Rel}(\mathrm{Map}(\mathcal{R}))$.

Remarks:

- Tabularity is a factorization condition: every morphism factors uniquelty as a left adjoint after a right adjoint.
- We sketch cartesian bicategories below.
- The theorem actually extends to a 2-equivalence between 2-categories (as you would expect).

Cartesian bicategories

Order-enriched symmetric monoidal categories

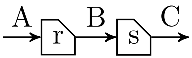
In an order-enriched category has a poset $\mathcal{A}(A, B)$ for objects A and B. Composition is order-preserving.

In the symmetric monoidal stucture $(\otimes, \mathbb{I}, ...)$, $\otimes$ is order-preserving in both arguments.

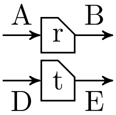
Wiring diagrams

Morphisms can be depicted as wiring diagrams:

- Composition $r; s$:

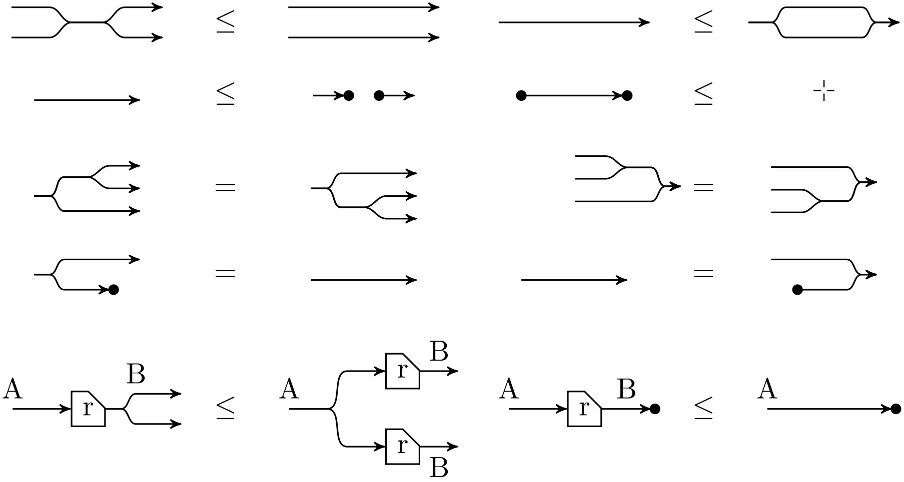


- Monoidal product $r \otimes t$:



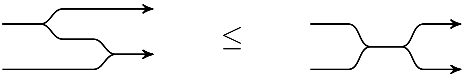
Wiring diagrams for cartesian bicategories

Every object is equipped with morphisms (δ_A split; $\delta_A^\dagger$ fuse; κ_A terminate; $\kappa_A^\dagger$ start)



Frobenius and categories of ordinary relations

In Rel($\mathcal{A}$) each object satisfies



Theorem
A tabulated cartesian bicategory is a category of relations iff the Frobenius axiom holds.

Remark
Rel(Pos), Rel(SL), etc., fail. For the order-enriched setting, we need alternatives to regularity and to Frobenius.

Ordinary versus Order Regular Categories

Order-regular category (Kurz & Velebil)

An order-enriched category $\mathcal{A}$ is order-regular iff

- $\mathcal{A}$ has all finite weighted limits
- $\mathcal{A}$ has order-strong/order-mono factorization
- order-strong morphisms are stable under pullbacks

* Strong morphisms are "effective" — the order version of regular epi

Idea

Order-regularity really is regularity suitably adapted:

- All finite limits means all finite limits including weighted ones
- Subobjects are order-embeddings
- Strong morphisms are defined with respect to these subobjects

Relations in order-enriched categories

Order composition

Replace pullback in the definition with comma:

Composition $\varphi \mathbin{⨟} \psi$ is defined by

- Pick tabulations $A \xleftarrow{p_R} R \xrightarrow{q_R} B$ and $B \xleftarrow{p_S} S \xrightarrow{q_S} C$ of φ and ψ
- form a comma (r, s) of (q_R, p_S).
- $\varphi \mathbin{⨟} \psi$ is the relation $[p_R r, q_S s]$.

Remarks

- Basic composition (; defined by pullback) still makes sense.
- Obviously $\varphi ; \psi \leq \varphi \mathbin{⨟} \psi$
- Not as obviously, $\varphi ; (\psi \mathbin{⨟} \theta) = (\varphi ; \psi) \mathbin{⨟} \theta$.
- $\mathbin{⨟}$ does not have identity basic relations: $\Delta_A \mathbin{⨟} \Delta_A \neq \Delta_A$.

2^{nd} Justifying Theorem

Theorem

For an order-enriched category $\mathcal{A}$ with finite weighted limits and order-strong/order-mono factorization the following are equivalent:

1. Basic composition (;) is associative.
2. Strong morphisms are stable under pullback

Morover, if these hold, then order composition ($\fatsemi$) is also associative.

Remarks on the proof

- The proof of (1) $\Leftrightarrow$ (2) is directly analogous to the ordinary situation.
- Associativity of order composition requires a technical lemma about pasting pullback and comma squares

Order relations

Identities

$$\begin{aligned} \Delta_A &= \text{the diagonal relation on } A \\ \mathbb{1}_A &= \Delta_A \fatsemi \Delta_A \end{aligned}$$

Order relations

For basic relation φ t.f.a.e.

$$\begin{aligned} \Delta_A \fatsemi \varphi &\leq \varphi \\ \mathbb{1}_A ; \varphi &\leq \varphi \\ \mathbb{1}_A \fatsemi \varphi &\leq \varphi \end{aligned}$$

Likewise for $\varphi \fatsemi \ldots$

An order relation is a basic relation satisfying $\mathbb{1}_A \fatsemi \varphi \fatsemi \mathbb{1}_B \leq \varphi$.

Two relation categories

Basic and order relation categories

- $\mathrm{bRel}(\mathcal{A})$: objects of $\mathcal{A}$, basic relations, basic composition, and Δ_A for identity.
- $\mathrm{oRel}(\mathcal{A})$: objects of $\mathcal{A}$, order relations, order composition, and $\mathbb{1}_A$ for identity.

Remarks

- $\Delta_A \mathbin{⨾} \varphi \mathbin{⨾} \Delta_B$ is smallest order relation contining φ.
- $\mathrm{bRel}(\mathcal{A})(A, B)$ and $\mathrm{oRel}(\mathcal{A})(A, B)$ are closed under finite meets (defined by pullback).
- $\Delta_A \leq \mathbb{1}_A$,
  $\mathbb{1}_A \mathbin{⨾} \mathbb{1}_A \leq \mathbb{1}_A$, and
  $\mathbb{1}_A \wedge \mathbb{1}_A^\circ \leq \Delta_A$.
- For order relations $\varphi \mathbin{⨾} \psi \leq \varphi ; \psi$

From $\mathcal{A}$ to $\mathrm{oRel}(\mathcal{A})$

Lemma

In an order regular category, for any cospan $A \xrightarrow{h} C \xleftarrow{k} B$, the corresponding comma tabulates an order relation.

Functors $f \mapsto \hat{f}$ and $f \mapsto \check{f}$

- Let $\hat{f}$ be the order relation tabulated by the comma of $A \xrightarrow{f} B \xleftarrow{\mathrm{id}_B} B$
- Let $\check{f}$ be the order relation tabulated by the comma of $B \xrightarrow{\mathrm{id}_B} B \xleftarrow{f} A$

Not hard to check:

$$\mathbb{1}_A \leq \hat{f} \,;\, \check{f} \qquad \text{and} \qquad \check{f} \,;\, \hat{f} \leq \mathbb{1}_B.$$

And $f \mapsto \hat{f}$ and $f \mapsto \check{f}$ are functorial.

$\mathrm{Map}(\mathrm{oRel}(\mathcal{A}))$

Theorem

For order relations φ and ψ in an order regular category, if

$$\mathbb{1}_A \leq \varphi \fatsemi \psi \qquad \text{and} \qquad \psi \fatsemi \varphi \leq \mathbb{1}_B,$$

then there is a unique $f{:}A \to B$ so that $\varphi = \hat{f}$ (and $\psi = \check{f}$).

$\mathrm{Map}(\mathrm{oRel}(\mathcal{A}))$

Consists of adjoint pairs (φ, ψ) of order relations, ordered by the right adjoint, taking domain and codomain from the left adjoint.

Theorem

For any order regular category $\mathcal{A}$, $\mathrm{Map}(\mathrm{oRel}(\mathcal{A}))$ is equivalent to $\mathcal{A}$.

The other way around

Recall

Theorem
A tabulated cartesian bicategory is a category of relations iff the Frobenius axiom holds.

We want a similar characterization for $\mathrm{oRel}(\mathcal{A})$

But simply dropping the Frobenius axiom is not quite enough.

Not quite

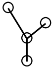
Lemma

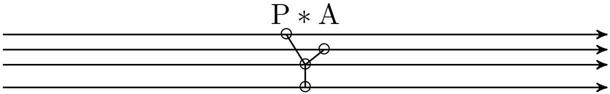
For any tabular cartesian bicategory $\mathcal{R}$, the order-enriched category $\mathrm{Map}(\mathcal{R})$ has all finite conical limits, has order-strong/order-mono factorization, and order-strong morphisms are stable under pullback.

So, existence of non-conical limits is all we are missing.

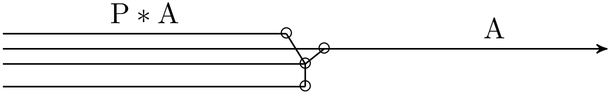
Ordered wires

For a finite poset P and object A, have an object $\mathrm{P} * \mathrm{A}$ (a single “structured” wire).

For example, for poset  and object A, we have an object.



Also, have $\hat{\omega}{:}\mathrm{P} \to \mathcal{R}(\mathrm{P} * \mathrm{A}, \mathrm{A})$, natural in P so each $\hat{\omega}_{\mathrm{i}}$ is a left adjoint



ordered in the obvious way. For an order preserving function $\mathrm{s}{:}\mathrm{P} \to \mathcal{R}(\mathrm{A}, \mathrm{B})$, there is a morphism $(\!|\mathrm{s}|\!){:}\mathrm{A} \to \mathrm{P} * \mathrm{B}$. These satisfy various coherence axioms, e.g, $(\!|\mathrm{s}|\!) ; \omega_{\mathrm{i}} = \mathrm{s}_{\mathrm{i}}$, also expressed in terms of ordered wires.

Wrapping up

Theorem

Suppose $\mathcal{R}$ is a tabulated cartesian bicategory, and each A is equipped with

- a functor $- * \mathrm{A}{:}\mathrm{Pos}_\omega^{\mathrm{op}} \to \mathcal{R}$
- $\hat{\omega}{:}\mathrm{P} \to \mathcal{R}(\mathrm{P} * \mathrm{A}, \mathrm{A})$ natural in P
- $(\!| - |\!){:}\mathrm{Pos}(\mathrm{P}, \mathcal{R}(\mathrm{A}, \mathrm{B})) \to \mathcal{R}(\mathrm{A}, \mathrm{P} * \mathrm{B})$ natural in P

Then $\mathrm{Map}(\mathcal{R})$ is order-regular iff these data satisfy

$$(\!|\hat{\omega}|\!) = \mathrm{id}_{\mathrm{P}*\mathrm{A}}$$

and

$$(\!|\mathrm{s}|\!); \hat{\omega}_\mathrm{i} = \mathrm{s}_\mathrm{i}$$

Moreover, then $\mathcal{R} \equiv \mathrm{oRel}(\mathrm{Map}(\mathcal{R}))$.

Thank you