Paths without homotopy, and homotopy without paths

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"Samson Abramsky on Logic and Structure in Computer Science and Beyond"

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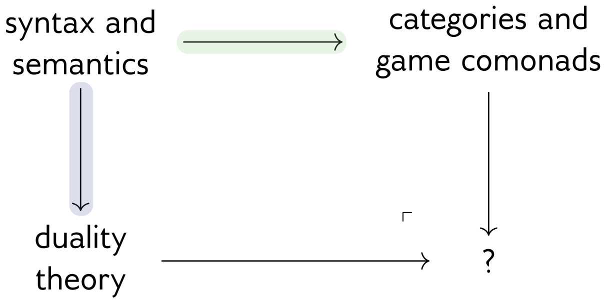
Logic and Structure



Gehrke, Jakl & LR. *A Cook's tour of duality in logic: From quantifiers, through Vietoris, to measures.* OCL 25, Springer 2023.

Abramsky & LR. *Arboreal categories: An axiomatic theory of resources.* ICALP'21, LMCS 2023.
Abramsky & LR. *Arboreal categories and homomorphism preservation theorems.*

From Concrete to Abstract: Game comonads

The Pebbling Comonad in Finite Model Theory

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***Abstract*—Pebble games are a powerful tool in the study of finite model theory, constraint satisfaction and database theory. Monads and comonads are basic notions of category theory which are widely used in semantics of computation and in modern functional programming. We show that existential k-pebble games have a natural comonadic formulation. Winning strategies for Duplicator in the k-pebble game for structures A and B are equivalent to morphisms from A to B in the coKleisli category for this comonad. This leads on to comonadic characterisations of a number of central concepts in Finite Model Theory:**

One of the key tools in studying these notions is that of existential k-pebble games [6]. Such a game, for structures A, B, proceeds by Spoiler placing one of his k pebbles on an element of the universe of A. Duplicator then places one of her pebbles on an element of B. If Duplicator is always able to move so that the partial mapping from A to B defined by sending a_i, the element in A carrying the i'th Spoiler pebble, to b_i, the corresponding element of B carrying the i'th Duplicator pebble, is a homomorphism on the induced substructures, then Duplicator has a winning strategy.

Proposition 1 ([6]). *The following are equivalent:*

Moving from the concrete setting of games to the abstract one of game comonads, we can

- better identify structure and patterns
- isolate the aspects that are specific to the context from the "context-free" ones

- Pebble comonad (Abramsky, Dawar & Wang, 2017)
- Ehrenfeucht-Fraïssé and modal comonads (Abramsky & Shah, 2018)
- Hella comonad (Ó Conghaile & Dawar, 2021)
- Guarded comonad (Abramsky & Marsden, 2021)
- Pebble-relation comonad (Montacute & Shah, 2022)
- Hybrid comonad (Abramsky & Marsden, 2022)

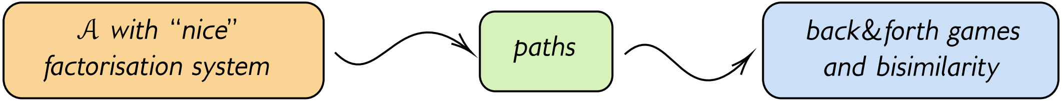
From Abstract to Axiomatic: Arboreal categories

A number of features appear to be common to all game comonads.

Can we reason about a generic game comonad?

... what is a *game comonad* ?

(Abramsky & LR, 2021) Arboreal categories as an axiomatic approach to game comonads:



The axioms for arboreal categories ensure that back-and-forth equivalence coincides with open-map bisimilarity in the sense of (Joyal, Nielsen & Winskel, 1993).

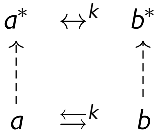
Homomorphism Preservation Theorems

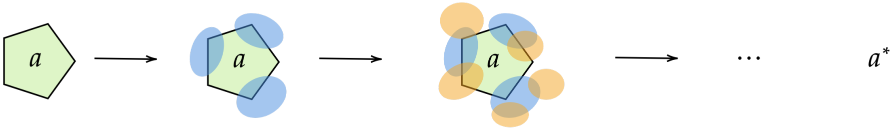
At the axiomatic level of arboreal categories we can establish axiomatic proofs of equi-resource Homomorphism Preservation Theorems, such as:

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

The key idea is that of upgrading: given structures a, b, construct extensions a^*, b^* such that a^* and b^* are FO_k-equivalent whenever a and b are $\exists^+\mathrm{FO}_k$-equivalent.





This is akin to a small object argument in (abstract) homotopy theory.

Upgrading

Upgrading arguments as the previous one are pervasive in (finite) model theory, see e.g.

Otto, *Model-theoretic methods for fragments of FO and special classes of (finite) structures.* In: "Finite and Algorithmic Model Theory", Cambridge University Press, 2011.

The general idea is that of constructing an "extension" of a given structure that preserves certain prescribed properties and, in addition, is symmetrical or "saturated". Cf. e.g. the construction of saturated elementary extensions in classical model theory. From the viewpoint of homotopy theory, this can be thought of as a form of fibrant replacement.

Can we make this analogy precise?

A (naive?) attempt

In the axiomatic setting, equivalence of structures in various logic fragments can be captured by spans of open morphisms.

Is there a “homotopy structure” in which open morphisms = weak equivalences ?

$$\nexists$$

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Weak Factorizations, Fractions and Homotopies

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Abstract. We show that the homotopy category can be assigned to any category equipped with a weak factorization system. A classical example of this construction is the stable category of modules. We discuss a connection with the open map approach to bisimulations proposed by Joyal, Nielsen and Winskel.

Mathematics Subject Classifications (2000): 18A99, 55P10, 68Q85.

Key words: weak factorization system, homotopy, category of fractions, bisimilarity.

Quillen model categories

A model category is a category $\mathcal{X}$ equipped with three classes of morphisms

$$\begin{aligned} \mathcal{W} &: \text{weak equivalences} \\ \mathcal{F} &: \text{fibrations} \\ \mathcal{C} &: \text{cofibrations} \end{aligned}$$

satisfying the following properties:

1. $\mathcal{X}$ is (finitely) complete and cocomplete;
2. $\mathcal{W}$ has the two-out-of-three property;
3. The pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorisation systems on $\mathcal{X}$.

For the duality theorists in the room

$$\text{sets} \ : \ \text{categories} \ = \ \text{topological spaces} \ : \ (\infty, 1)\text{-categories}$$

Quillen model categories are presentations of $(\infty, 1)$-categories.

Joyal's Proposition E.1.10

A model structure is determined by its cofibrations together with its class of fibrant objects.

A model category for modal logic

Let $\mathbf{K}$ be the category of Kripke models. For each $n \leq \omega$, the tree unravelling up to level n gives a coreflection

$$\mathbf{S}_n \underset{R_n}{\overset{\top}{\hookrightarrow}} \mathbf{K}$$

of $\mathbf{K}$ into the full subcategory $\mathbf{S}_n$ consisting of synchronization trees of height at most n.
For all $a \in \mathbf{K}$, a and $R_n a$ satisfy the same modal formulas with modal depth $\leq n$.

- $\mathbf{P}_n$: subcategory of $\mathbf{S}_n$ whose objects are the synchronization trees with a single branch, i.e. the traces, and whose morphisms are the embeddings.
- $\mathbf{S}_n^*$: wide subcategory of $\mathbf{S}_n$ whose morphisms (preserve and) reflect the unary relations. The restricted Yoneda embedding gives $\mathbf{S}_n^* \hookrightarrow \widehat{\mathbf{P}}_n$.

The presheaf category $\widehat{\mathbf{P}}_n$ admits a Quillen model structure such that any $X \xrightarrow{f} Y$ in $\mathbf{S}_n^*$

- is an embedding just when it is a cofibration.
- is a p-morphism (i.e., bounded) just when it is a trivial fibration.

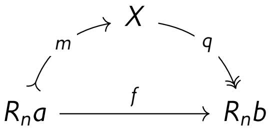
A preservation theorem for modal logic

Theorem (Andréka, van Benthem & Németi; Rosen)

A modal formula of depth $\leq n$ is preserved under embeddings of Kripke models iff it is equivalent to an existential modal formula of depth $\leq n$ (constructed from the atoms and their negations, using $\wedge, \vee$ and $\Diamond$). Further, the result relativises to finite Kripke models.

Proof. Let φ be a modal formula of depth n preserved under embeddings, and let $a, b \in \mathbf{K}$ satisfy $a \Rrightarrow_n^{\exists} b$. We must prove that $a \models \varphi \;\Rightarrow\; b \models \varphi$. Equivalently, $R_n a \models \varphi \;\Rightarrow\; R_n b \models \varphi$.

The condition $a \Rrightarrow_n^{\exists} b$ implies the existence of a morphism $f\colon R_n a \to R_n b$ in $\mathbf{S}_n^*$. Now, there is a $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ factorisation of f in $\widehat{\mathbf{P}}_n$ such that $X \in \mathbf{S}_n^*$:



So m is an embedding and q is a p-morphism. Thus, $R_n a \models \varphi \;\Rightarrow\; R_n b \models \varphi$.

Further, if a, b are finite, so is X. □

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