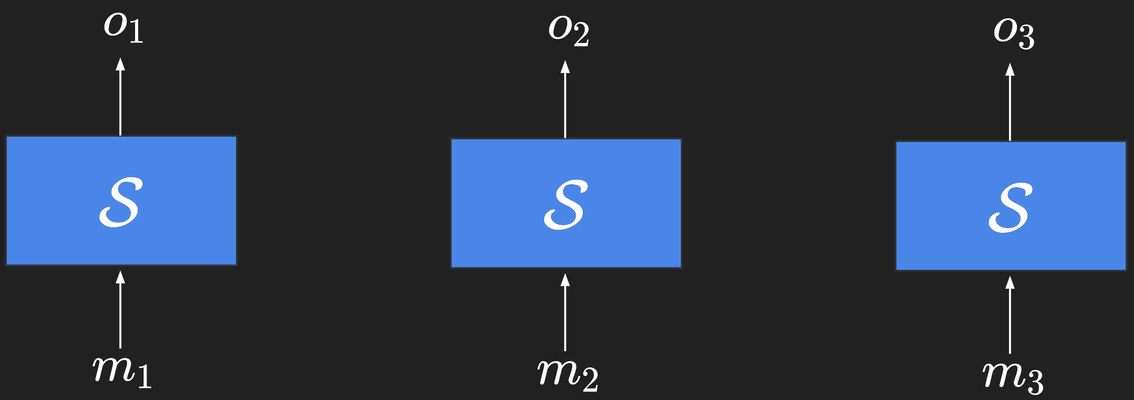
Quantifying Non-Classicality in Temporal Measurement Scenarios

Samson Abramsky, Rui Soares Barbosa, Amy Searle

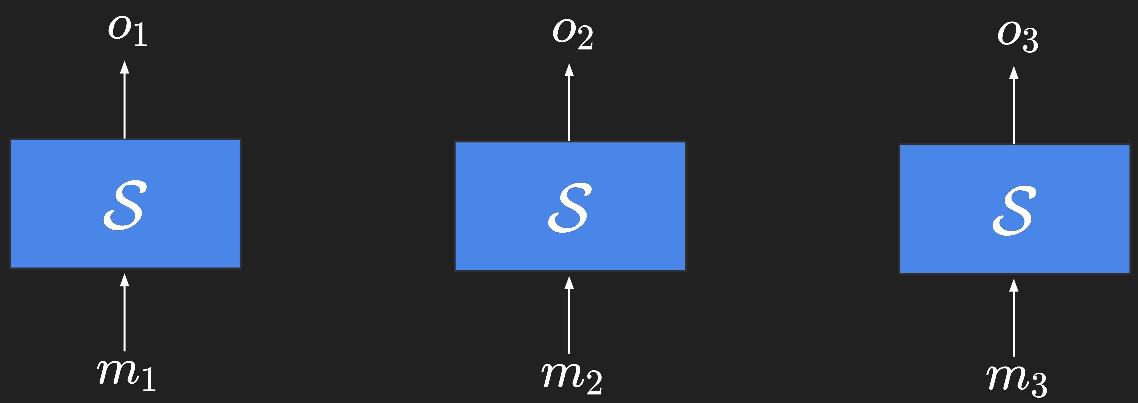
Part 1: The Framework

Setup



Setup

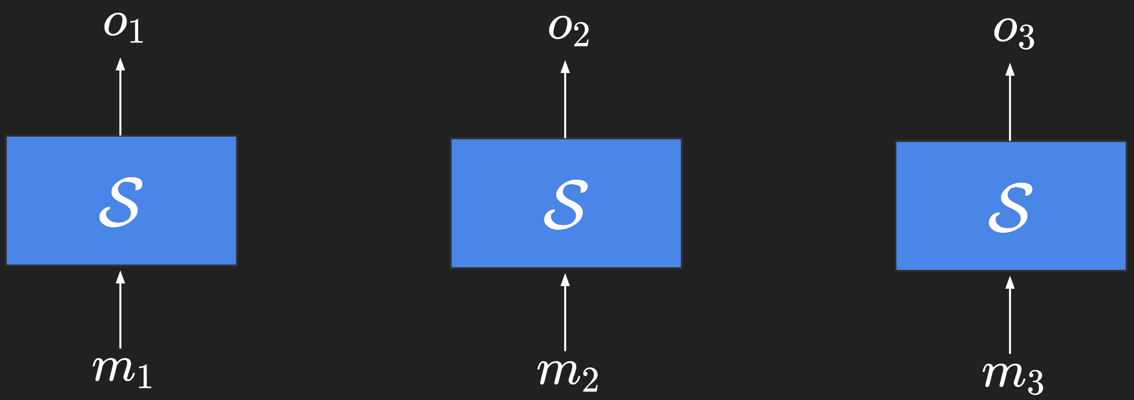
X : Set of measurements



Setup

X : Set of measurements

O : Set of outcomes for each measurement

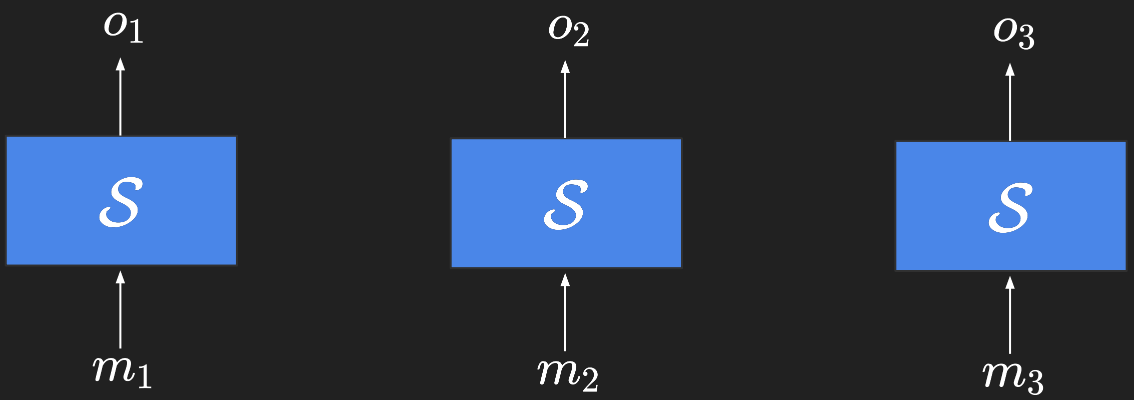


Setup

X : Set of measurements

O : Set of outcomes for each measurement

X^* : Set of all possible sequences of measurements in X



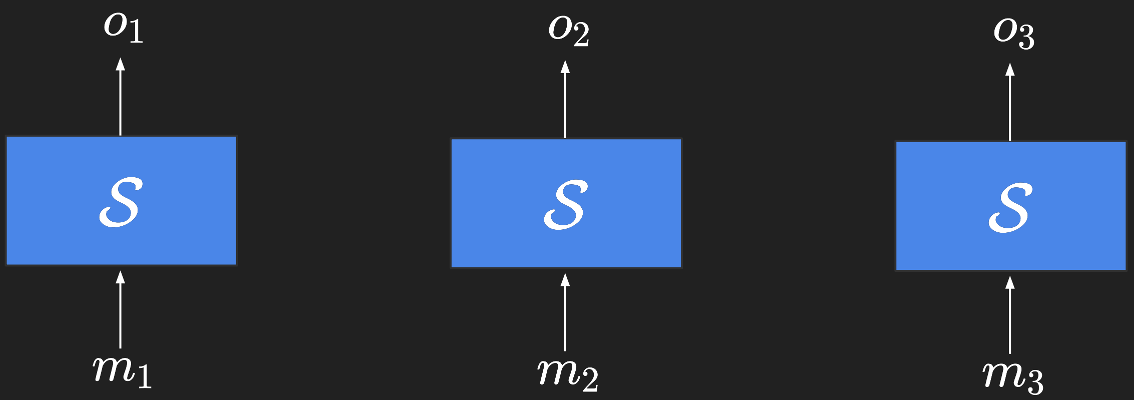
Setup

X : Set of measurements

O : Set of outcomes for each measurement

X^* : Set of all possible sequences of measurements in X

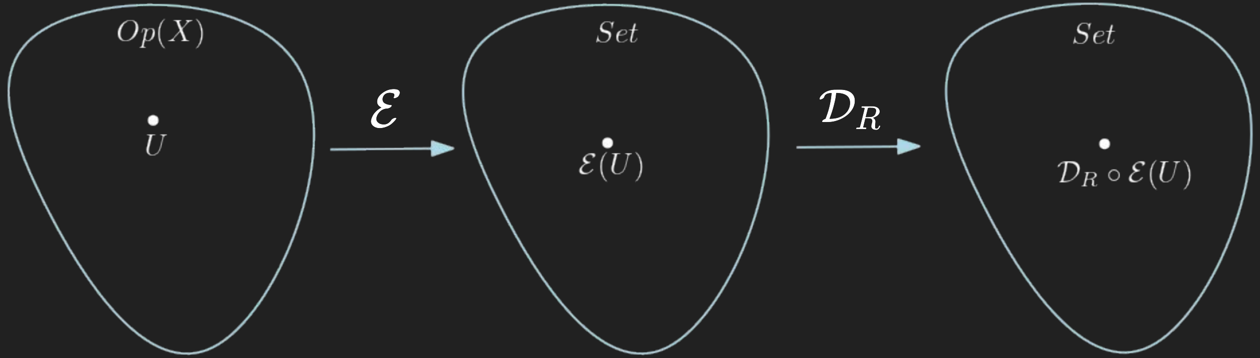
Σ : A left-closed subset of X^*



Motivation

- What constitutes as a *classical system* in this setup is not well understood
- Casting in a sheaf theoretic framework allows us to use tools from sheaf theory
- The sheaf theoretic framework for *spatial* correlations has included
  - Logical Bell Inequalities
  - Contextual Fraction
  - Notions of Simulation
  - Cohomological obstructions to classicality

Presheaves– Notation



$U, U' \ldots$: Sets of observables

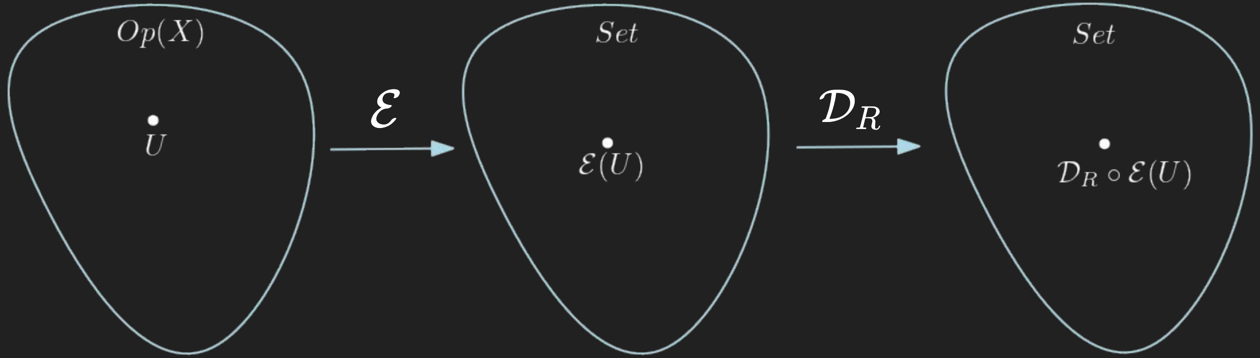
$\mathcal{E}(U), \mathcal{E}(U') \ldots$

: Mapping observables to outcomes

$\mathcal{D}_R(\mathcal{E}(U)), \mathcal{D}_R(\mathcal{E}(U')) \ldots$

: Probability distributions over these mappings

Presheaves– Notation



$U, U' \ldots$: Sets of observables

$x \in U$:
Measurement, Set of Measurements

$\mathcal{E}(U), \mathcal{E}(U') \ldots$
: Mapping observables to outcomes

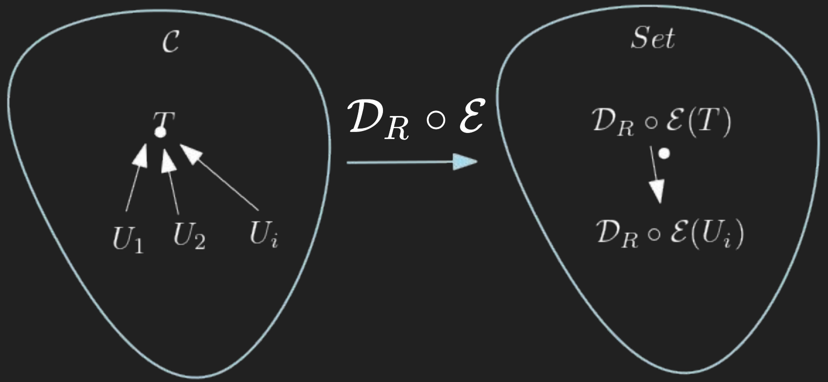
$f \in \mathcal{E}(U)$: Behaviour of the System

$\mathcal{D}_R(\mathcal{E}(U)), \mathcal{D}_R(\mathcal{E}(U')) \ldots$
: Probability distributions over these mappings

$e_U \in \mathcal{D}_R \circ \mathcal{E}(U)$:
Probabilistic Behaviour of the System

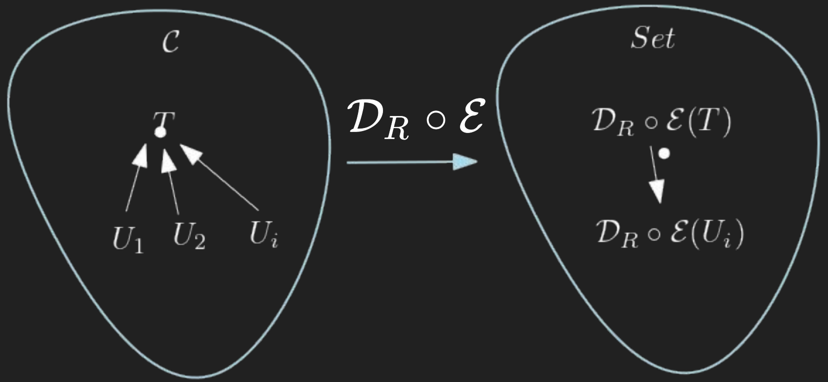
Sheaves

A presheaf is a sheaf when the gluing and uniqueness axioms are satisfied:



Sheaves

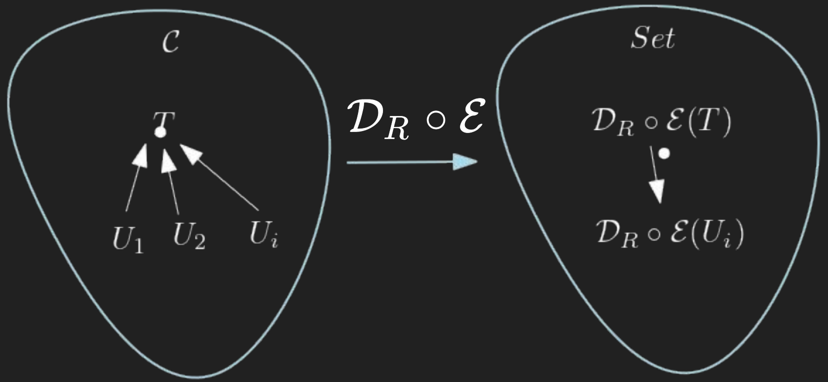
A presheaf is a sheaf when the gluing and uniqueness axioms are satisfied:



Cover $\{U_i\}_{i \in I}$

Sheaves

A presheaf is a sheaf when the gluing and uniqueness axioms are satisfied:

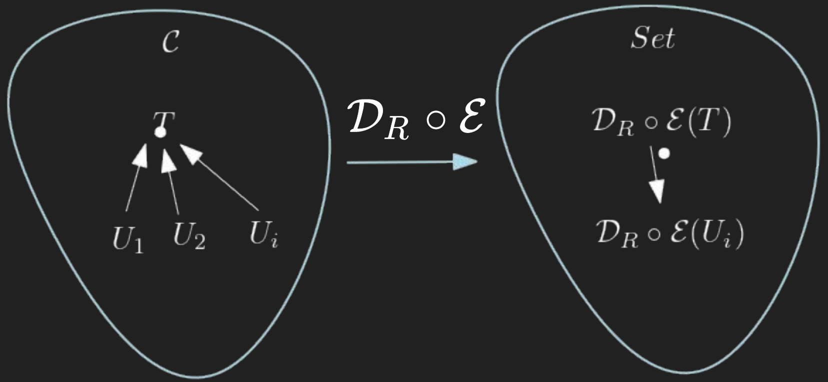


Cover $\{U_i\}_{i \in I}$

Given data $\{e_i \in \mathcal{D}_R \circ \mathcal{E}(U_i)\}_{i \in I}$

Sheaves

A presheaf is a sheaf when the gluing and uniqueness axioms are satisfied:



Cover $\{U_i\}_{i \in I}$

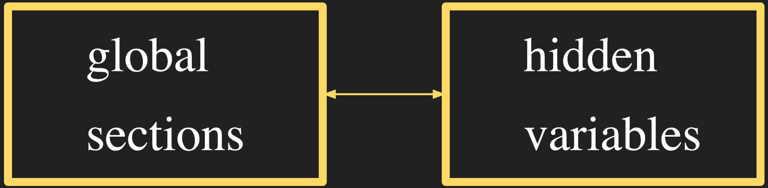
Given data $\{e_i \in \mathcal{D}_R \circ \mathcal{E}(U_i)\}_{i \in I}$

Exists a global section $h \in \mathcal{D}_R \circ \mathcal{E}(T)$ with $h|_{U_i} = e_i$

Sheaves and Classicality

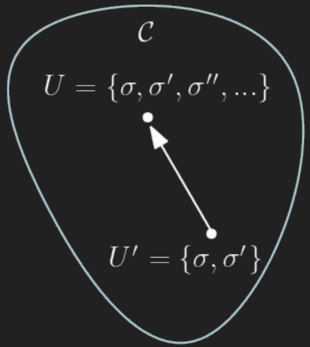
Proposition 3.1. [Abramsky, Brandenburger]:

The existence of a global section for an empirical model implies the existence of a local (or non-contextual) deterministic hidden-vairable model which realizes it.



A Sheaf Approach To Temporal Correlations

Step 1: Define the category $\mathcal{C}$



Objects: Down closed subsets of Σ

Arrows: Subset inclusion

Step 2: Define the map $\mathcal{E} : \mathcal{C} \to Set$

$$\mathcal{E}(U) = \{f : U \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$

The Important Bits

Empirical Models

$$\{e_\sigma \in \mathcal{D}_R \circ \mathcal{E}(\sigma)\}_{\sigma \in \Sigma}$$

The Important Bits

Empirical Models

$$\{e_\sigma \in \mathcal{D}_R \circ \mathcal{E}(\sigma)\}_{\sigma \in \Sigma}$$

Classicality

$$h \in \mathcal{D}_R \circ \mathcal{E}(\Sigma) \text{ with } h|_\sigma = e_\sigma \text{ for all } \sigma \in \Sigma$$

Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(\Sigma) = \{f : \Sigma \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$

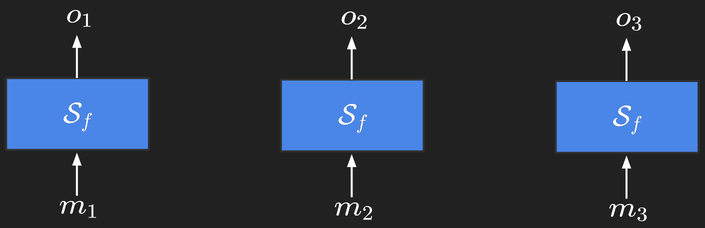
Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(\Sigma) = \{f : \Sigma \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$



$$f((m_1)) = o_1$$
$$f((m_1, m_2)) = o_2$$
$$f((m_1, m_2, m_3)) = o_3$$

Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(\Sigma) = \{f : \Sigma \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$

$$f((m_1)) = o_1$$
$$f((m_1, m_2)) = o_2$$
$$f((m_1, m_2, m_3)) = o_3$$

$$f((m_1)) = o_1$$
$$f((m_1, m'_2)) = o'_2$$
$$f((m_1, m'_2, m_3)) = o'_3$$

Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(\Sigma) = \{f : \Sigma \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$

$f((m_1)) = o_1$
$f((m_1, m_2)) = o_2$
$f((m_1, m_2, m_3)) = o_3$

$$k = 0 \Rightarrow o_3 = o_3'$$

$f((m_1)) = o_1$
$f((m_1, m_2')) = o_2'$
$f((m_1, m_2', m_3)) = o_3'$

Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(\Sigma) = \{f : \Sigma \to O :: \sigma \mapsto o\}$$

$$\text{lookback}_k(\sigma) = \text{lookback}_k(\sigma') \Rightarrow f(\sigma) = f(\sigma')$$

$f((m_1)) = o_1$

$f((m_1, m_2)) = o_2$

$f((m_1, m_2, m_3)) = o_3$

$$k = 0 \Rightarrow o_3 = o'_3$$

$$k > 0, \text{ can have } o_3 \neq o'_3$$

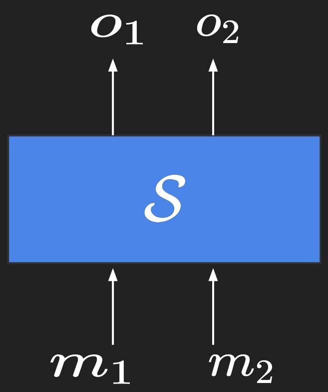
$f((m_1)) = o_1$

$f((m_1, m'_2)) = o'_2$

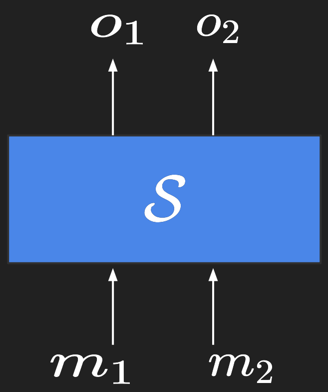
$f((m_1, m'_2, m_3)) = o'_3$

Part 2: The Mapping

Contextuality Setup

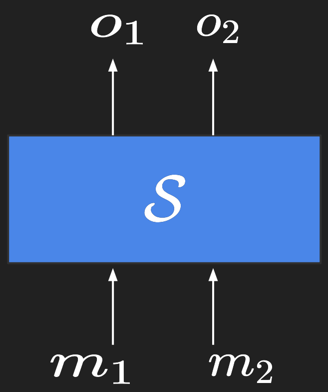


Contextuality Setup



X : Set of measurements

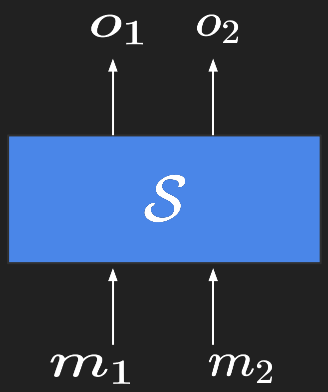
Contextuality Setup



X : Set of measurements

O : Set of outcomes for each measurement

Contextuality Setup

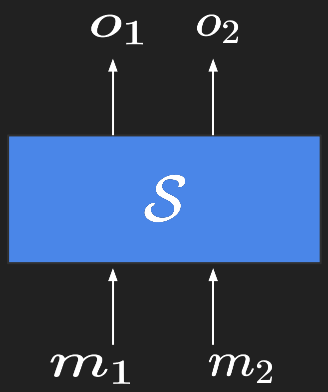


X : Set of measurements

O : Set of outcomes for each measurement

Σ : Measurement Contexts

Contextuality Setup



X : Set of measurements

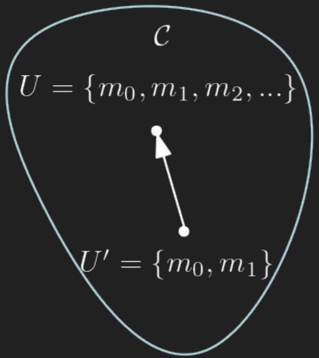
O : Set of outcomes for each measurement

Σ : Measurement Contexts

Note: Σ is a simplicial complex in this case

A Sheaf Approach to Contextuality Correlations

Step 1: Define the category $\mathcal{C}$



Objects: Subsets of the measurement set X

Arrows: Subset inclusion

Step 2: Define the map $\mathcal{E} : \mathcal{C} \to Set$

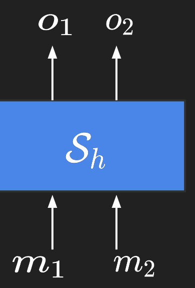
$$\mathcal{E}(U) = O^U = \{s : U \to O :: m \mapsto o\}$$

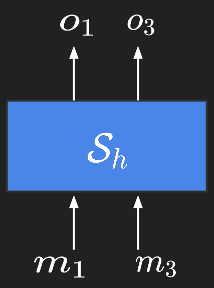
Classicality

What does it mean to be classical?

1. Deterministic Classicality

$$\mathcal{E}(X) = \{h : X \to O :: m \mapsto o\}$$





$$h(m_1) = o_1, h(m_2) = o_2$$

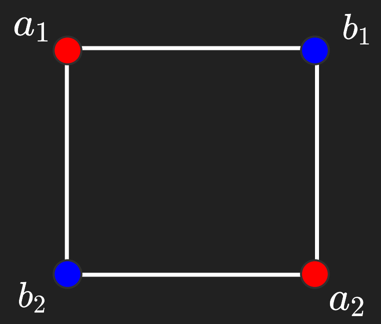
$$h(m_1) = o_1, h(m_3) = o_3$$

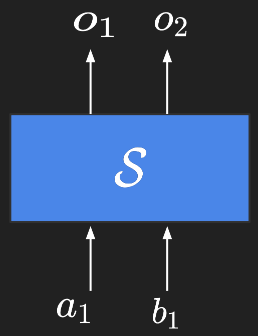
Vorob'ev's Theorem for Contextuality Setups

Proposition [Vorobev, Barbosa]:

Let Σ be a simplicial complex. Then any empirical model defined on Σ is extendable if and only if Σ is acyclic

Example: CHSH



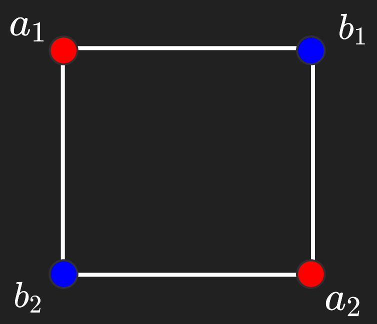


Vorob'ev's Theorem for Contextuality Setups

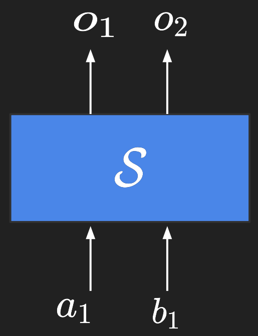
Proposition [Vorobev, Barbosa]:

Let Σ be a simplicial complex. Then any empirical model defined on Σ is extendable if and only if Σ is acyclic

Example: CHSH



Not acyclic so there can be a contextual empirical model!



Mapping Scenarios for $k = \infty$

$$\mathcal{C}_\infty : \mathbf{EM}(\mathcal{M}) \rightarrow \mathbf{EM}(\mathcal{M}')$$

$1. X' := \Sigma$

$2. U' \in \Sigma' \iff \cup_{\sigma \in U'} \sigma \in \Sigma$

$3. O_{(m_0, m_1, \ldots, m_k)} := O_{m_k}$

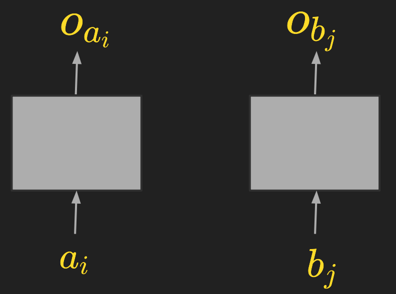
Mapping Scenarios for $k = \infty$

$$\mathcal{C}_\infty : \mathbf{EM}(\mathcal{M}) \to \mathbf{EM}(\mathcal{M}')$$

$1. X' := \Sigma$

$2. U' \in \Sigma' \iff \cup_{\sigma \in U'} \sigma \in \Sigma$

$3. O_{(m_0, m_1, \ldots, m_k)} := O_{m_k}$



Example:

$X' = \{(a_1, b_1), (a_1, b_2),$
$(a_2, b_1), (a_2, b_2), (a_1), (a_2)\}$

$\{(a_1), (a_1, b_1)\} \in \Sigma'$

$O_{(a_1, b_1)} = O \times O = \{(0,0), (0,1), (1,0), (1,1)\}$

Mapping Scenarios for $k = \infty$

$$\mathcal{C}_\infty : \mathbf{EM}(\mathcal{M}) \to \mathbf{EM}(\mathcal{M}')$$

1. $X' := \Sigma$

2. $U' \in \Sigma' \iff \bigcup_{\sigma \in U'} \sigma \in \Sigma$

3. $O_{(m_0, m_1, \ldots, m_k)} := O_{m_k}$

Example:

$X' = \{(a_1, b_1), (a_1, b_2),$
$(a_2, b_1), (a_2, b_2), (a_1), (a_2)\}$

$\{(a_1), (a_1, b_1)\} \in \Sigma'$

$O_{(a_1, b_1)} = O \times O = \{(0,0), (0,1), (1,0), (1,1)\}$

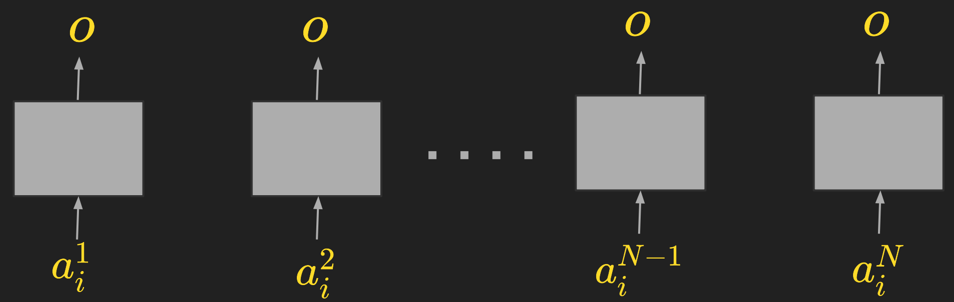
Theorem 3.2. *For lookback depth $k = \infty$, there is a map $\mathscr{C}_\infty$ from temporal measurement scenarios to contextuality measurement scenarios, which lifts to empirical models as a convex function $\mathscr{C}_\infty : \mathsf{EM}_\infty \to \mathsf{EM}(\mathscr{C}_\infty(M))$ mapping each ∞-lookback empirical model e on the temporal scenario $\mathscr{M}$, to an empirical model $\mathscr{C}_k(e)$ on the measurement scenario $\mathscr{C}_\infty(\mathscr{M})$. This map preserves and reflects nonclasicality, meaning that an empirical model e on $\mathscr{M}$ is not ∞-lookback classical if and only if $\mathscr{C}_\infty(e)$ is contextual.*

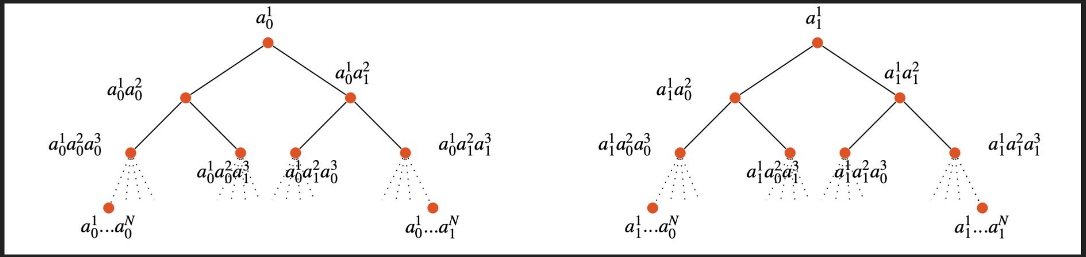
No Quantum Advantage for $k = \infty$

Proposition: Given any temporal measurement scenario $\mathcal{M}$ every empirical model $e \in EM(\mathcal{M})$ is ∞-lookback classical.

No Quantum Advantage for $k = \infty$

Proposition: Given any temporal measurement scenario $\mathcal{M}$ every empirical model $e \in EM(\mathcal{M})$ is ∞-lookback classical.





A (Still Speculative) More General Mapping

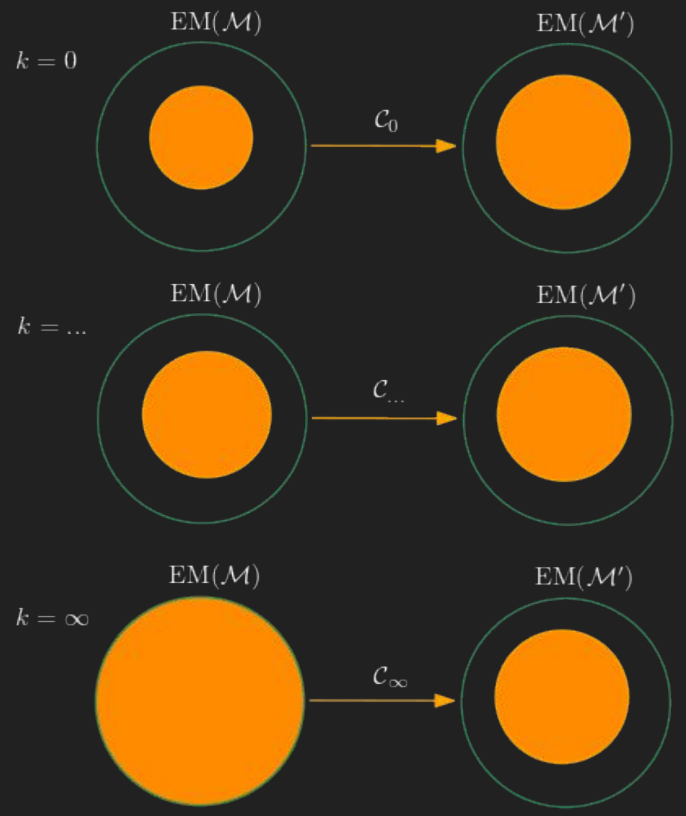
$$\mathcal{C}_k : \mathbf{EM}(\mathcal{M}) \to \mathbf{EM}(\mathcal{M}')$$

$$1. X' := \{(m, h) | m \in X, \exists \sigma = (\ldots, m) \in \Sigma.\, h \in \mathrm{lookback}_k(\sigma)\}$$

$$2. U' \in \Sigma' \iff \exists \sigma \in \Sigma. \forall (m, (m_0, \ldots, m_l)) \in U'.\, (m_0, \ldots, m_l, m) \subseteq \sigma$$

$$3. O_{(m,h)} := O_m$$

Visualising this Mapping



References

[1] S. Abramsky and A. Brandenburger, "The sheaf-theoretic structure of non-locality and contextuality," *New Journal of Physics*, vol. 13, p. 113036, Nov. 2011.

[2] S. Gogioso and N. Pinzani, "The sheaf-theoretic structure of definite causality," *Electronic Proceedings in Theoretical Computer Science*, vol. 343, pp. 301–324, sep 2021.

[3] A. J. Leggett and A. Garg, "Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?," *Phys. Rev. Lett.*, vol. 54, pp. 857–860, Mar 1985.

[4] T. Fritz, "Quantum correlations in the temporal Clauser-Horne-Shimony-Holt (CHSH) scenario," *New Journal of Physics*, vol. 12, p. 083055, Aug. 2010.

[5] N. N. Vorob'ev, "Consistent Families of Measures and Their Extensions," *Theory of Probability Its Applications*, vol. 7, no. 2, pp. 147–163, 1962.

[6] R. Barbosa, "Contextuality in quantum mechanics and beyond," 2015.

[7] C. Budroni, "Contextuality, memory cost and non-classicality for sequential measurements," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 377, no. 2157, p. 20190141, 2019.

[8] L. B. Vieira and C. Budroni, "Temporal correlations in the simplest measurement sequences," *Quantum*, vol. 6, p. 623, jan 2022.

[9] S. Mansfield, "A unified approach to contextuality and violations of macrorealism.." https://www1.psych.purdue.edu/ ehtibar/workshop/files/leacht$_{p}rague.pdf, June$2017.