Mixed Distributive Laws

Game Comonads over the Quantum Monad

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The Setting

Let σ be a set of relational symbols with positive arities, we can define a category of σ-structures $\mathscr{R}(\sigma)$:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R \in \sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

Setting for graph theory, database theory, and descriptive complexity

For simplicity, we will take σ to have one binary relation R.

Two games

Spoiler vs. Duplicator in the one-sided k-pebble Duplicator from $\mathcal{A}$ to $\mathcal{B}$

- Spoiler moves around k-many pebbles on vertices of $\mathcal{A}$ picking out window.
- Duplicator responds with placing corresponding pebbles on $\mathcal{B}$.
- Duplicator continues to not lose if the relation induced by pair of windows is a partial homomorphism from $\mathcal{A} \rightharpoonup \mathcal{B}$.

Two games

Verifier vs. Alice+Bob in a non-local game

- A+B want to convince a Verifier that there is morphism $\mathcal{A} \to \mathcal{B}$
- Verifier sends a pair of vertices $(a, a') \in \mathcal{A}$.
- Alice+Bob send vertices $(b, b') \in \mathcal{B}$.
- A+B win if relation $R(a, a') \Rightarrow R(b, b')$.

A+B winning is equivalent to a classical homomorphism $\mathcal{A} \to \mathcal{B}$. To obtain a notion of quantum homomorphism:

- A+B perform measurements (via POVMs $\{A_{a,b}\}, \{B_{a',b'}\}$) on an entangled state in $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$.
- A+B win *perfectly* if $p(b, b'|a, a') = \psi^*(A_{a,b} \otimes B_{a',b'})\psi = 1$
- Quantum perfect strategies yield a definition of quantum homomorphism.
- There are cases, e.g. Mermin's magic square, where A+B win with this quantum advantage, but lose classically.

Two constructions

The Pebbling Comonad in Finite Model Theory by Abramsky, Dawar, and Wang [2]

- Comonad family $(\mathbb{P}_k, \delta, \varepsilon)$ on $\mathscr{R}(\sigma)$ w/ inclusions $\mathbb{P}_{k'} \hookrightarrow \mathbb{P}_k$ for $k' \leq k$
- $\mathbb{P}_k(\mathcal{A}) \to \mathcal{B}$ correspond to Duplicator winning strategies in the one-sided k-pebble game.
- Formalised tacit connections between k-consistency test, full/counting k-variable logic, treewidth $< k$, k-Weisfeiler-Leman test

The Quantum Monad on Relational Structures by Abramsky, Barbosa, de Silva, and Zapata [1]

- Graded monad $(\mathsf{Q}_d, \mu^{d,d'}, \eta_1)$ on $\mathscr{R}(\sigma)$.
- $\mathcal{A} \to \mathsf{Q}_d(\mathcal{B})$ correspond to quantum perfect winning strategies in a non-local game.
- Connections to contextuality and quantum advantage in algorithms for CSPs.

Pebbling and EF comonads

$\mathbb{P}_k(\mathcal{A})$ is set of k-pebble plays:

$$s = [(p_1, a_1), \dots, (p_n, a_n)]$$

Counit $\varepsilon_{\mathcal{A}} : \mathbb{P}_k(\mathcal{A}) \to \mathcal{A}$ sends s to a_n.

Comultiplication $\delta_{\mathcal{A}} : \mathbb{P}_k(\mathcal{A}) \to \mathbb{P}_k(\mathbb{P}_k(\mathcal{A}))$ sends s to $[(p_1, s_1), \dots, (p_n, s_n)]$ where $s_i = [(p_1, s_1), \dots, (p_i, s_i)]$.

$(s, t) \in R^{\mathbb{P}_k(\mathcal{A})}$ if $s \sqsubseteq t$ or $t \sqsubseteq s$, an "active pebble" condition, and $(\varepsilon(s), \varepsilon(t)) \in R^{\mathcal{A}}$.

$\mathbb{E}_k(\mathcal{A})$ is set of k-length plays:

$$s = [a_1, \dots, a_n] \quad n \leq k$$

Counit, comultiplication defined similarly as for $\mathbb{P}_k$.

In definition of $R^{\mathbb{E}_k(\mathcal{A})}$, we drop the active pebble condition.

Quantum monad

Modification of probability distribution monad with probability distributions replaced with projector-valued measurements.

A 'distribution monad' over the partial commutative semiring of $\mathbf{Proj}(d)$.

$\mathsf{Q}_d(\mathcal{A})$ is a set of mappings $p\colon A \to \mathbf{Proj}(d)$ which we can satisfy a normalization condition:

$$\sum_{a\in A} p(a) = 1$$

and have finite support.

Pairwise orthogonal: For all $a, a' \in \mathrm{supp}(p)$, $p(a)p(a') = p(a')p(a) = 0$.

An element $p\colon A \to \mathbf{Proj}(d) \in \mathsf{Q}_d(\mathcal{A})$ can be written as formal sum:

$$\sum_{a\in A} p(a).a$$

Quantum monad (cont.)

$R^{\mathbf{Q}_d(\mathcal{A})}$ be the set of tuples (p_1, p_2) satisfying:

- $[p_1(a_1), p_2(a_2)] = \mathbf{0}$ for all $a_1, a_2 \in A$
- if $(a_1, a_2) \notin R^{\mathcal{A}}$, then $p_1(a_1)p_2(a_2) = \mathbf{0}$

Unit $\eta_{\mathcal{A}} : A \to \mathbf{Q}_1\mathcal{A}$ sends a to the 'dirac delta distribution' on a, i.e. $\eta(a) = I_1.a$

Multiplication $\mu_{\mathcal{A}}^{d,d'} : \mathbf{Q}_d\mathbf{Q}_{d'}\mathcal{A} \to \mathbf{Q}_{dd'}\mathcal{A}$

$$\mu_{\mathcal{A}}^{d,d'}(P)(a) = \sum_{p \in \mathbf{Q}_{d'}(\mathcal{A})} P(p) \otimes p(a)$$

Written as a formal sum:

$$\mu_{\mathcal{A}}^{d,d'}(P) = \sum_{a \in A} \sum_{p \in \mathbf{Q}_{d'}(\mathcal{A})} P(p) \otimes p(a).a$$

Question

Both these constructions provide a clean presentation composing strategies in the games they 'internalise'

Given morphisms $f\colon \mathbb{P}_{k'}(\mathcal{A}) \to \mathcal{B}$ and $g\colon \mathbb{P}_k(\mathcal{B}) \to \mathcal{C}$ for $k' \leq k$, we obtain

$$\mathbb{P}_{k'}(\mathcal{A}) \hookrightarrow \mathbb{P}_k(\mathcal{A}) \xrightarrow{\delta_{\mathcal{A}}} \mathbb{P}_k(\mathbb{P}_k(\mathcal{A})) \xrightarrow{\mathbb{P}_k(f)} \mathbb{P}_k(\mathcal{B}) \xrightarrow{g} \mathcal{C}$$

Given morphisms $h\colon \mathcal{A} \to \mathsf{Q}_d(\mathcal{B})$ and $k\colon \mathcal{B} \to \mathsf{Q}_{d'}(\mathcal{C})$, we obtain

$$\mathcal{A} \xrightarrow{f} \mathsf{Q}_d(\mathcal{B}) \xrightarrow{\mathsf{Q}_d(g)} \mathsf{Q}_d(\mathsf{Q}_{d'}(\mathcal{C})) \xrightarrow{\mu_{\mathcal{C}}^{d,d'}} \mathsf{Q}_{dd'}(\mathcal{C})$$

Morphisms of type $\mathbb{P}_k(\mathcal{A}) \to \mathsf{Q}_d(\mathcal{B})$ encode Duplicator strategies with partial *quantum* homomorphisms as the winning condition.

How to compose morphisms of type $\mathbb{P}_k(\mathcal{A}) \to \mathsf{Q}_d(\mathcal{B})$?

Mixed distributive laws

From natural transformations $\kappa^d \colon \mathbb{P}_k \circ \mathsf{Q}_d \to \mathsf{Q}_d \circ \mathbb{P}_k$, we could obtain:

$$\mathbb{P}_{k'}\mathcal{A} \hookrightarrow \mathbb{P}_k\mathcal{A} \xrightarrow{\delta} \mathbb{P}_k\mathbb{P}_k\mathcal{A} \xrightarrow{\mathbb{P}_k(f)} \mathbb{P}_k\mathsf{Q}_d\mathcal{B} \xrightarrow{\kappa_{\mathcal{B}}} \mathsf{Q}_d\mathbb{P}_k\mathcal{B} \xrightarrow{\mathsf{Q}_d(g)} \mathsf{Q}_d\mathsf{Q}_{d'}\mathcal{C} \xrightarrow{\mu^{d,d'}} \mathsf{Q}_{dd'}\mathcal{C}$$

from $f \colon \mathbb{P}_{k'}(\mathcal{A}) \to \mathsf{Q}_d(\mathcal{B})$ and $g \colon \mathbb{P}_k(\mathcal{B}) \to \mathsf{Q}_{d'}(\mathcal{C})$ for $k' \leq k$.

This composition in $\mathscr{R}(\sigma)$ yields a composition in $\mathbf{biKl}(\mathbb{P}_k, \mathsf{Q}_d)$ if κ satisfies the equations:

$$\kappa^1 \circ \mathbb{P}_k\eta = \eta\mathbb{P}_k \quad \kappa^{dd'} \circ \mathbb{P}_k\mu^{d,d'} = \mu^{d,d'}\mathbb{P}_k \circ \mathsf{Q}_d\kappa^d \circ \kappa^{d'}\mathsf{Q}_{d'}$$

$$\mathsf{Q}_d\varepsilon \circ \kappa^d = \varepsilon\mathsf{Q}_d \quad \mathsf{Q}_d(\delta) \circ \kappa^d = \kappa\mathbb{P}_k \circ \mathbb{P}_k\kappa \circ \delta\mathsf{Q}_d$$

Does there exists a distributive law $\kappa^d \colon \mathbb{P}_k \circ \mathsf{Q}_d \to \mathsf{Q}_d \circ \mathbb{P}_k$?

Simplifying our question

Ignoring pebbles are there distributive laws of type:

$$\mathbb{E}\mathsf{Q}_d \to \mathsf{Q}_d\mathbb{E}?$$

Ignoring relations are the distributive laws for (co)monads on **Set** of type

$$L^+\mathsf{Q}_d \to \mathsf{Q}_d L^+?$$

Viewing the quantum monad 'at the level of probabilities or possiblities', laws of type:

$$L^+\mathcal{D} \to \mathcal{D}L^+?$$

$$L^+\mathcal{P}^+ \to \mathcal{P}^+L^+?$$

This last simplification turned out to be *wrong*, but led us to answer interesting, but unrelated questions.

There *are* distributive laws $\mathbb{E}\mathsf{Q}_d \to \mathsf{Q}_d\mathbb{E}$ and $\mathbb{E}_k\mathsf{Q}_d \to \mathsf{Q}_d\mathbb{E}_k$!

Proposition

There is a unique law $\kappa\colon L^+\mathcal{P}^+ \to L^+\mathcal{P}^+$ where

$$\kappa([X_1, \dots, X_n]) = \{[x_1, \dots, x_n] \mid \forall i \in \{1, \dots, n\}, x_i \in X_i\}$$

satisfying the unit axiom.

But... this unique κ does not satisfy the comultiplication axiom:

$$\mathcal{P}(\delta_X) \circ \kappa_X([\{a, b\}, \{c\}]) = \{[[a], [a, c]], [[b], [b, c]]\}$$
$$\kappa_{L^+X} \circ L^+(\kappa_X) \circ \delta_{\mathcal{P}(X)}([\{a, b\}, \{c\}]) = \{[[a], [a, c]], [[b], [b, c]], [[a], [b, c]], [[b], [a, c]]\}$$

Theorem

There is no distributive law of the prefix list comonad over the non-empty powerset monad.

We can generalise this theorem to a wider class of (co)monads:

Theorem

If the following hold:

- *W is directed container which has at least one non-root position.*
- *M is a distribution monad $\mathcal{D}_{\mathscr{S}}$ for some semiring $\mathscr{S}$ satisfying*
  - *$\mathscr{S}$ is zero-sum-free, i.e. for all $a, b \in \mathscr{S}$ if $a + b = 0$, then $a = 0$ and $b = 0$.*
  - *For some $n > 1$, $1_{\mathscr{S}} + 1_{\mathscr{S}} + \cdots + 1_{\mathscr{S}}$ (n times) is a unit.*

  Then there is no distributive law $\kappa\colon WM \to MW$.

Examples of directed containers W where our no-go theorem $\kappa \colon W\mathcal{D}_{\mathscr{S}} \to \mathcal{D}_{\mathscr{S}}W$ applies:

- Prefix list comonad L^+, Suffix tree comonad, Underlined list comonad
- Cowriter comonad $(\cdot)^M$ for monoid M

Non-Examples of W:

- Coreader comonads $S \times (\cdot)$
- Pointed powerset comonad $\mathcal{P}_*(X) = \{(Y, x) \mid x \in Y \subseteq X\}$

Examples/Non-examples

Examples of semirings $\mathscr{S}$ where our no-go theorem $\kappa\colon W\mathcal{D}_{\mathscr{S}} \to \mathcal{D}_{\mathscr{S}}W$ applies for $\mathcal{D}_{\mathscr{S}}$:

- $(\mathbb{B}, \vee, \wedge, \top, \bot)$ is the finite non-empty powerset monad
- $(\mathbb{R}_{\geq 0}, +, *, 0, 1)$ is the discrete probability distribution monad $\mathcal{D}$.
- Viterbi semiring $([0, 1], \max, *, 0, 1)$.

Non-examples of $\mathscr{S}$

- Any ring, e.g. $\mathbb{R}$
- $(\mathbb{N}, +, *, 0, 1)$
- For a fixed set T, $(\mathcal{P}(T), \cup, \cap, \varnothing, T)$

How do we extend our no-go theorem $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ on **Set** to $\kappa\colon \mathbb{E}_k\hat{\mathcal{P}} \to \hat{\mathcal{P}}\mathbb{E}_k$ on $\mathscr{R}(\sigma)$?

Transfer across categories

Theorem

Given $U\colon \mathscr{C} \to \mathscr{D}$, (co)monads $\hat{W}, \hat{M}$ on $\mathscr{C}$, (co)monads W, M on $\mathscr{D}$,

- $\hat{\kappa}\colon \hat{W}\hat{M} \to \hat{M}\hat{W}$, $\kappa\colon WM \to MW$ *nat. transformations*
- *General (co)Kleisli laws* $\theta_w\colon WU \to U\hat{W}$, $\theta_m\colon U\hat{M} \to MU$ *satisfying:*

$$
\begin{array}{ccccccc}
 & & U\hat{W}\hat{M} & \xrightarrow{U\hat{\kappa}} & U\hat{M}\hat{W} & & \\
 & \overset{\theta_w \hat{M}}{\nearrow} & & & & \overset{\theta_m \hat{W}}{\searrow} & \\
WU\hat{M} & & & & & & MU\hat{W} \\
 & \underset{W\theta_m}{\searrow} & & & & \underset{M\theta_w}{\nearrow} & \\
 & & WMU & \xrightarrow{\kappa U} & MWU & &
\end{array}
$$

Then

- *if κ is a dist. law and θ_w, θ_m have monic components, then $\hat{\kappa}$ is a dist law.*
- *if $\hat{\kappa}$ is a dist. law and θ_w, θ_m have epic components, then κ is a dist law.*

Generalises a result of Manes+Mulry [5] using the formal theory of Power+Watanbe [6]. Elegant string diagram proof!

Transfer with retraction

Theorem

If there exists a $\hat{\kappa}\colon \hat{W}\hat{M} \to \hat{M}\hat{W}$ *distributive law for comonads on* $\mathscr{C}$*,*

- $L \dashv U$ *a coreflection* $L\colon \mathscr{D} \to \mathscr{C}$ *and* $U\colon \mathscr{C} \to \mathscr{D}$
- *Componenet-wise split epimorphisms* $\theta_w\colon WU \to U\hat{W}$, $\theta_m\colon U\hat{M} \to MU$

Then there exists a nat. transformation $\kappa\colon WM \to MW$ *of (co)monads* W, M *on* $\mathscr{S}$ *which satisfies the Yang-Baxter equation, and thus a distributive law.*

Applying the contrapositive of this theorem and our no-go theorem, we obtain there is no distributive law of type $\mathbb{E}_k\hat{\mathcal{P}} \to \hat{\mathcal{P}}\mathbb{E}_k$ and $\mathbb{E}_k\hat{\mathcal{D}} \to \hat{\mathcal{D}}\mathbb{E}_k$ for any liftings $\hat{\mathcal{P}}, \hat{\mathcal{D}}\colon \mathscr{R}(\sigma) \to \mathscr{R}(\sigma)$

In particular, this holds for the tree-duality monad (a lifting of $\mathcal{P}$) and a monad capturing fractional isomorphism (a lifting of $\mathcal{D}$).

Aside: Transfer for other uses

Monad-monad version of this transfer with retraction theorem. Use this version to obtain a no-go theorem for a monad over **Top**.

The Vietoris monad $\mathsf{V}\colon \mathbf{Top} \to \mathbf{Top}$ sends a topological space (X, τ) to the 'hit-or-miss' topology on the set of closed subspaces $C \subseteq X$ of X.

There is a coreflection $L \dashv U$ with $U\colon \mathbf{Top} \to \mathbf{Set}$ forgetful, and $L\colon \mathbf{Set} \to \mathbf{Top}$ mapping a set X to it's discrete topology.

There is a component-wise split epimorphism $\theta_m \circ \theta_m^{-1}$ where

- $\theta_m\colon \mathcal{P}U \to U\mathsf{V}$ maps a subset $Y \subseteq X$ to is closure:

$$\langle Y \rangle = \cap\{C \mid Y \subseteq C, C \text{ is closed in } X\}$$

- $\theta_m^{-1}\colon U\mathsf{V} \to \mathcal{P}U$ maps a closed subset $C \subseteq_\tau X$ to its underlying set.

By a theorem of Klin+Salamanca [4] there is no distributive law of $\mathcal{P}\mathcal{P} \to \mathcal{P}\mathcal{P}$, so we obtain there is no distributive law $\mathsf{V}\mathsf{V} \to \mathsf{V}\mathsf{V}$.

A working distributive law

Define $\kappa^d_X \colon \mathbb{E}_k \mathbf{Q}_d \to \mathbf{Q}_d \mathbb{E}_k$ with components defined as:

$$\kappa_{\mathcal{A}}[\varphi_1, \ldots, \varphi_n] = \sum_{[a_1,\ldots,a_n] \in \mathbb{E}_k(\mathcal{A})} \varphi_1(a_1) \ldots \varphi_n(a_n).[a_1, \ldots, a_n]$$

Why does comultiplication not break as in the case $\mathcal{P}^+$?

Because of the structure of PVMs the extra ‘covariant’ terms are canceled out:

$$\begin{aligned}
\mathbf{Q}_d(\delta_A) \circ \kappa_A([Pa + Qb, Ic]) &= P[[a],[a,c]] + Q[[b],[b,c]] \\
\kappa_{\mathbb{E}_k A} \circ \mathbb{E}_k(\kappa_A) \circ \delta_{\mathbf{Q}_d(A)}([Pa + Qb, Ic]) &= P^2[[[a],[a,c]] + Q^2[[b],[b,c]] \\
&\quad + \color{red}{PQ[[a],[b,c]] + QP[[b],[a,c]]} \\
&= P^2[[a],[a,c]] + Q^2[[b],[b,c]] \\
&= P[[a],[a,c]] + Q[[b],[b,c]]
\end{aligned}$$

Follows from pairwise orthogonality $PQ = QP = 0$ and projector idempotence $P^2 = P, Q^2 = Q$.

Conclusion

No-Go theorems and the existence of a law $\kappa\colon \mathbb{E}_k\mathsf{Q}_d \to \mathsf{Q}_d\mathbb{E}_k$ suggest there are no possiblistic/probabilistic Duplicator winning strategies in the EF game, but there are quantum ones. Concrete construction?

Uniform 2-categorical proof of the Transfer Theorems? Application to other probability monads, e.g. Giry monad?

$\kappa\colon \mathbb{E}_k\mathsf{Q}_d \to \mathsf{Q}_d\mathbb{E}_k$ is a coKleisli law, so by work Jakl+Marsden+S [3], we obtain:

$$\mathcal{A} \equiv_{\exists^+\mathsf{FO}_k} \mathcal{B} \Rightarrow \mathsf{Q}_d(\mathcal{A}) \equiv_{\exists^+\mathsf{FO}_k} \mathsf{Q}_d(\mathcal{B})$$

$$\mathcal{A} \equiv_{\#\mathsf{FO}_k} \mathcal{B} \Rightarrow \mathsf{Q}_d(\mathcal{A}) \equiv_{\#\mathsf{FO}_k} \mathsf{Q}_d(\mathcal{B})$$

What about FO_k? Check (S1) and (S2) axioms in this paper.

Quantum k-consistency test for approximating quantum homomorphism?
Quantum k-Weisfeiler Leman test for approximating quantum isomorphism?

Connections with the local-global consistency in database theory?

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