Monads, Comonads, and transducers

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University College London

Samson Abramsky on Logic and Structure in Computer Science and Beyond

Founded by EPSRC project *“Resources in Computation”*

Monads

MA

Data structure

$MMA \longrightarrow MA$

Flattening operation

$A \longrightarrow MA$

Singleton operation

Monads

MA

Data structure

$MMA \longrightarrow MA$

Flattening operation

$A \longrightarrow MA$

Singleton operation

For example

$MA = A^*$

Monads

MA

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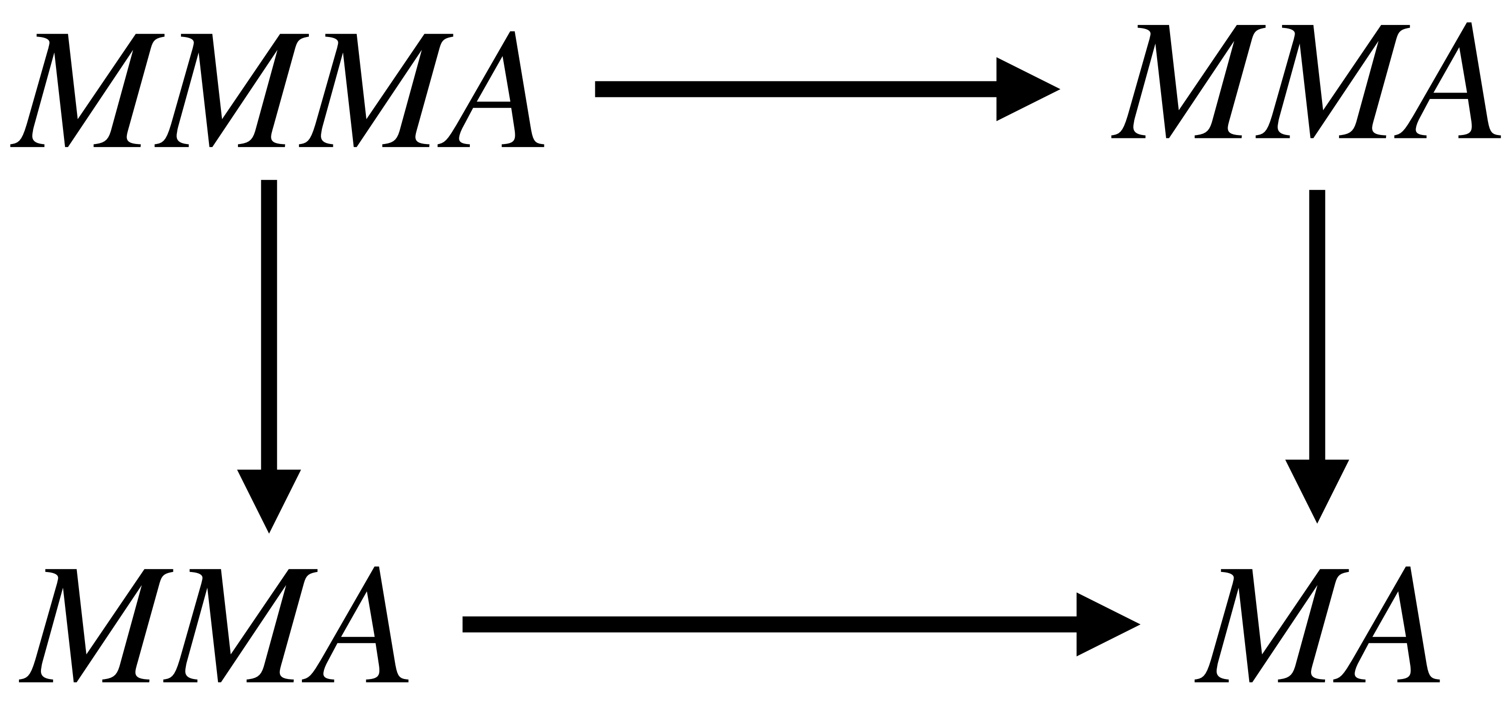
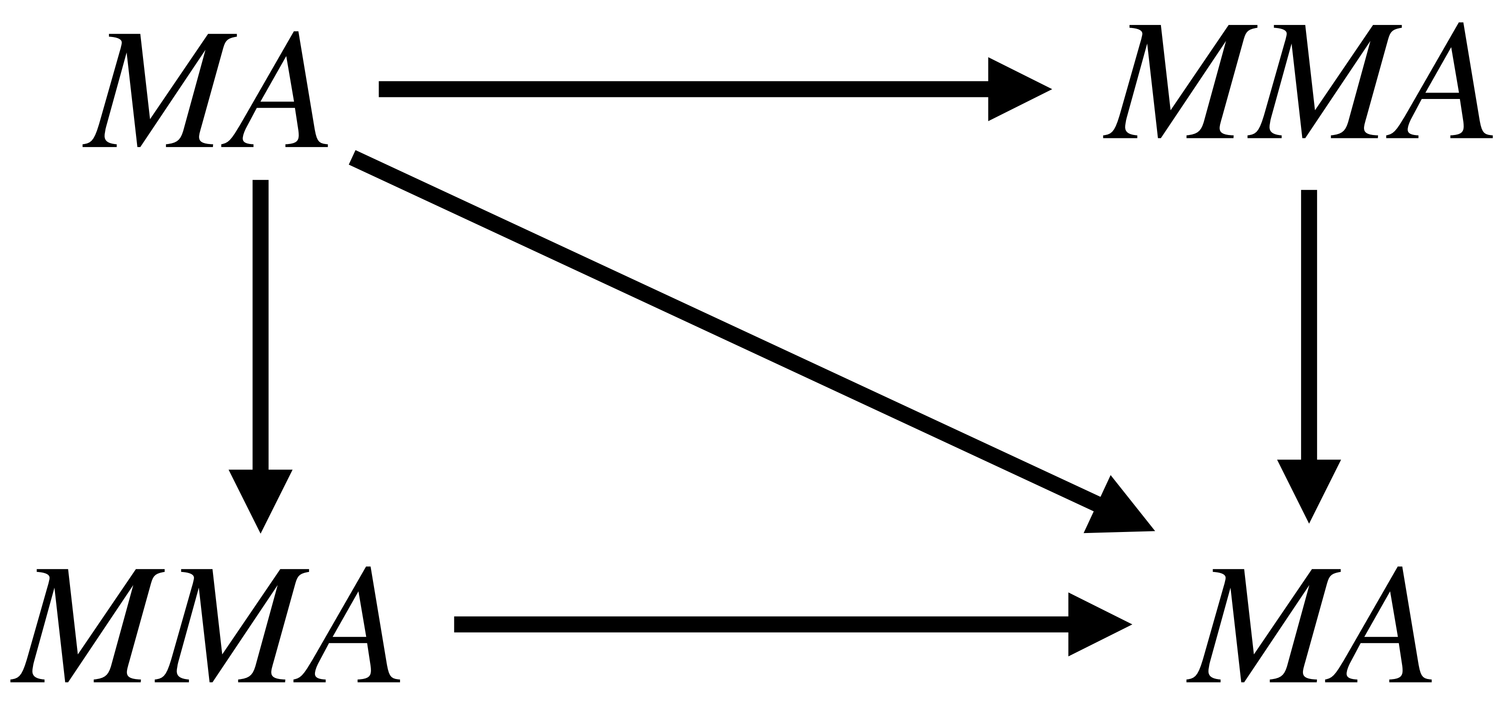
For example

$$MA = A^*$$

Together with coherence axioms





Comonads

MA

Data structure

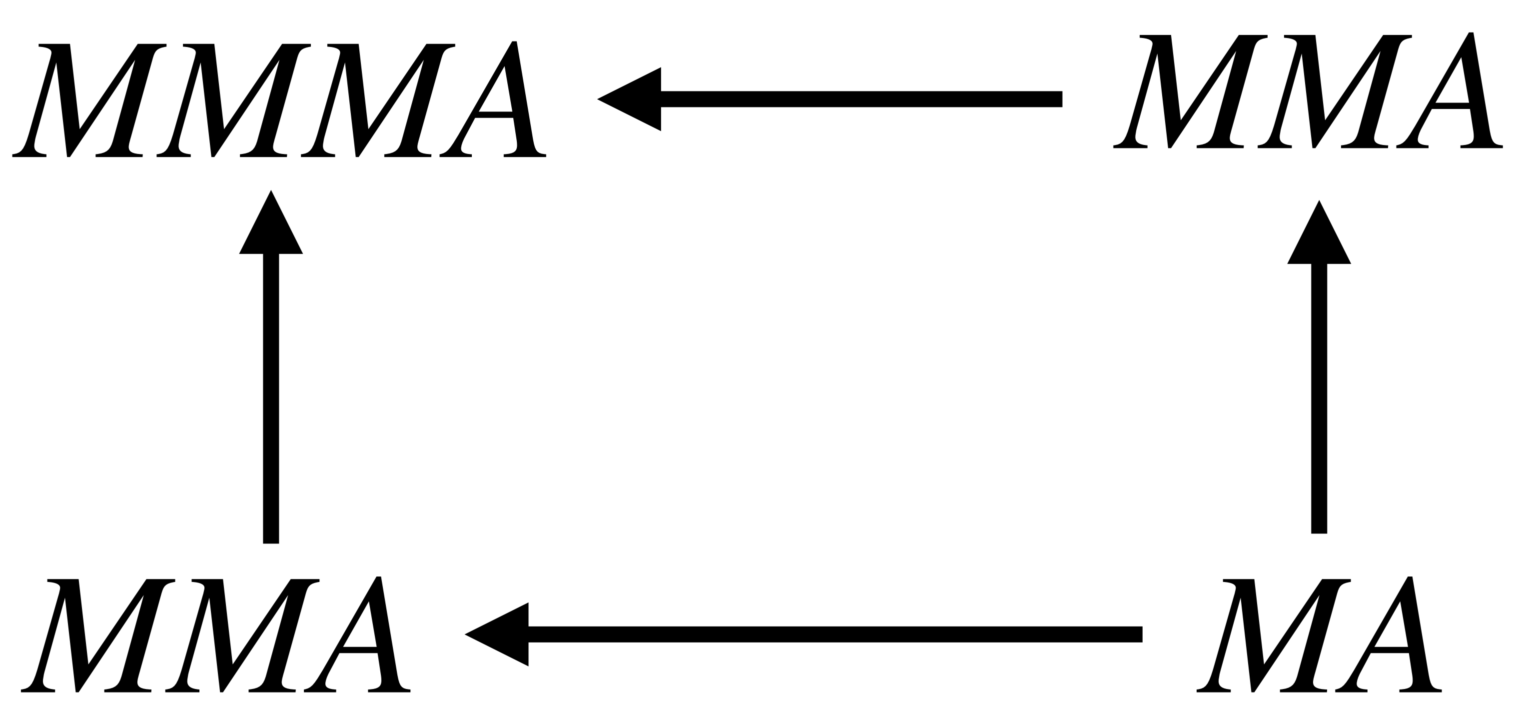
$MMA \longleftarrow MA$

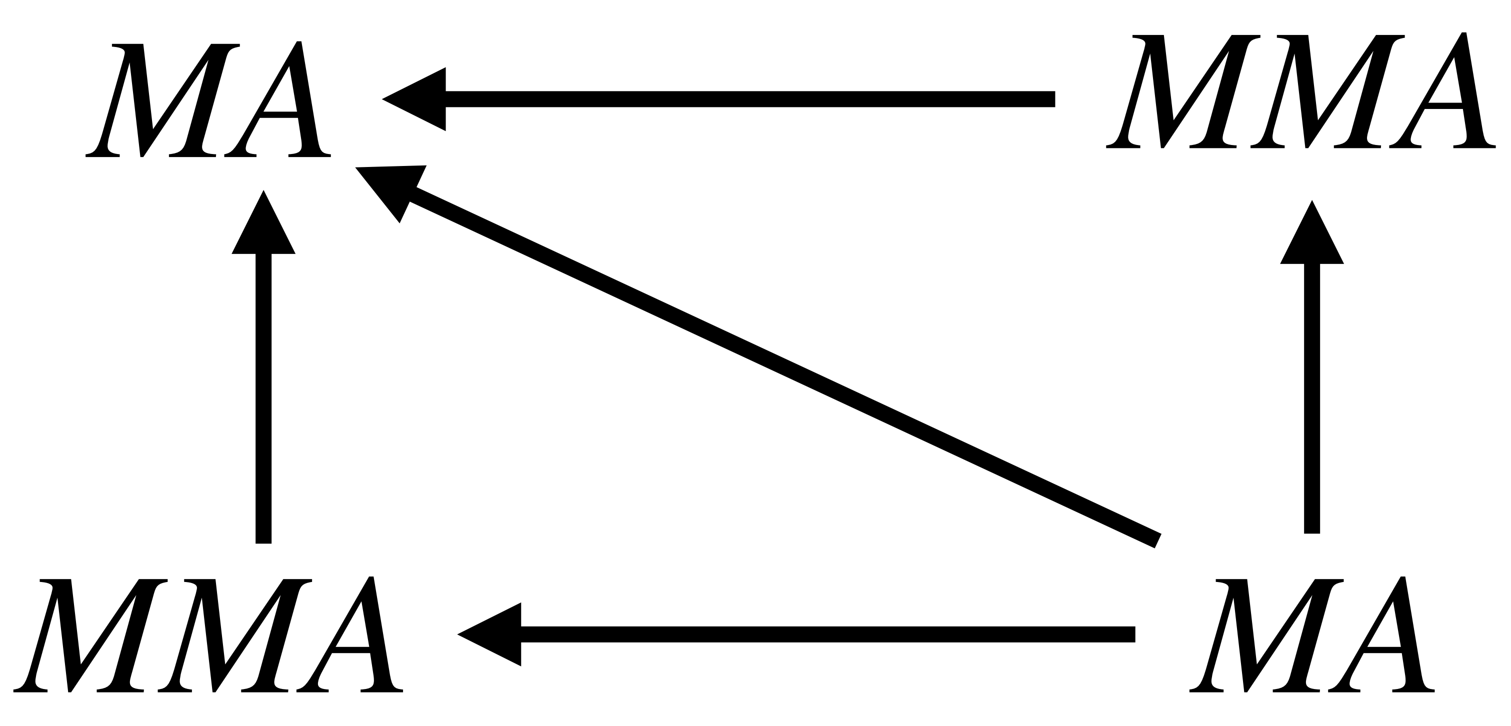
Expanding operation

$A \longleftarrow MA$

Extracting operation

Together with coherence axioms





Slogan:

Monads = Languages

Monads + Comonads = Transducers

Slogan:

Regular
=
Recognisable by finite algebras

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M. Bojańczyk. 2015.
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This talk.

Monad and comonad

$$MA = A^+$$

$MMA \to MA$

$A \to MA$

$MA \to MMA$

$MA \to A$

Monad and comonad

$$MA = A^+$$

$MMA \to MA$	Flatten	$[[1, 2, 3], [4, 5], [6, 7]] \mapsto [1,2,3,4,5,7]$
$A \to MA$	Singleton	$7 \mapsto [7]$

$$MA \to MMA$$

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$MA \to A$	Last element	

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$MA \to A$	Last element	$[1,2,3,4] \mapsto 4$

Monads, comonads, and transducers

Given a regular language:

$$L : M\Sigma \to \{\texttt{Yes}, \texttt{No}\}$$

We define the following transduction:

$$M\Sigma \xrightarrow{\text{comonad}} MM\Sigma \xrightarrow{ML} M\{\texttt{Yes}, \texttt{No}\}$$

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$[a, b, a, a]$

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$$M\Sigma \xrightarrow{\text{comonad}} MM\Sigma \xrightarrow{M\,L} M\Gamma$$

This gives us a class of M-transductions.

Structure vs. power

M	Expressive Power

Structure vs. power

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Non-empty lists with prefixes	Mealy machines

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Other examples of M:

Words over countable orders with a maximal/minimal/underlined element.

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...

Theorem

M-transductions are closed under compositions.

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$$\begin{array}{ccc} M\Sigma & \longrightarrow & M\Gamma \\ & & \downarrow \\ & & M\Delta \end{array}$$

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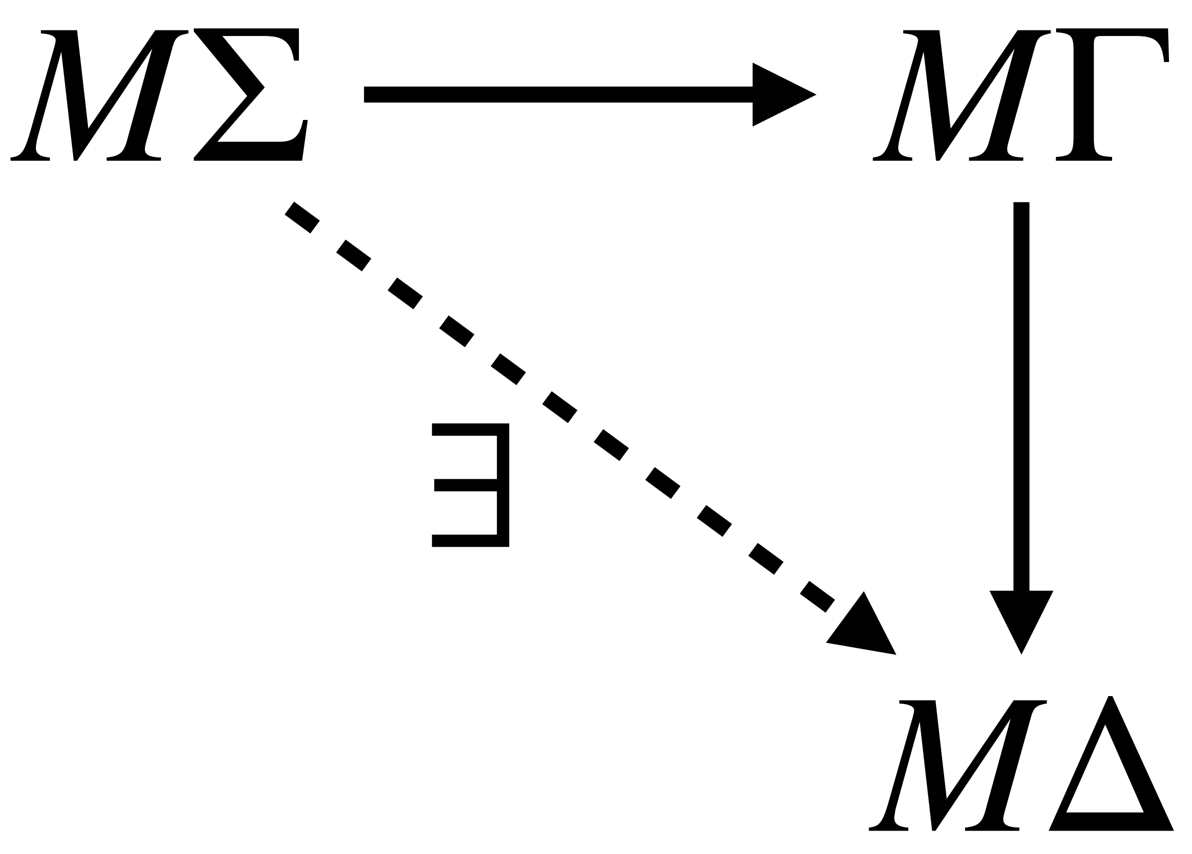

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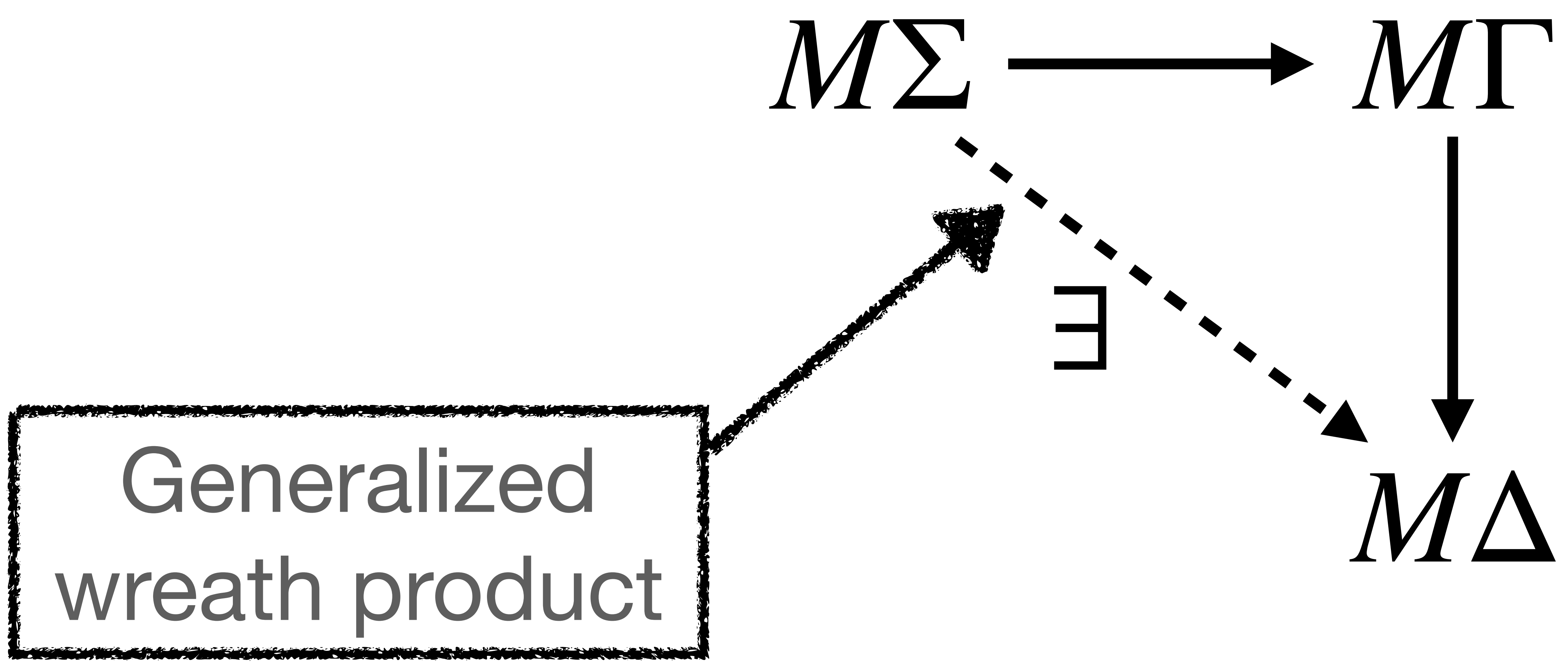

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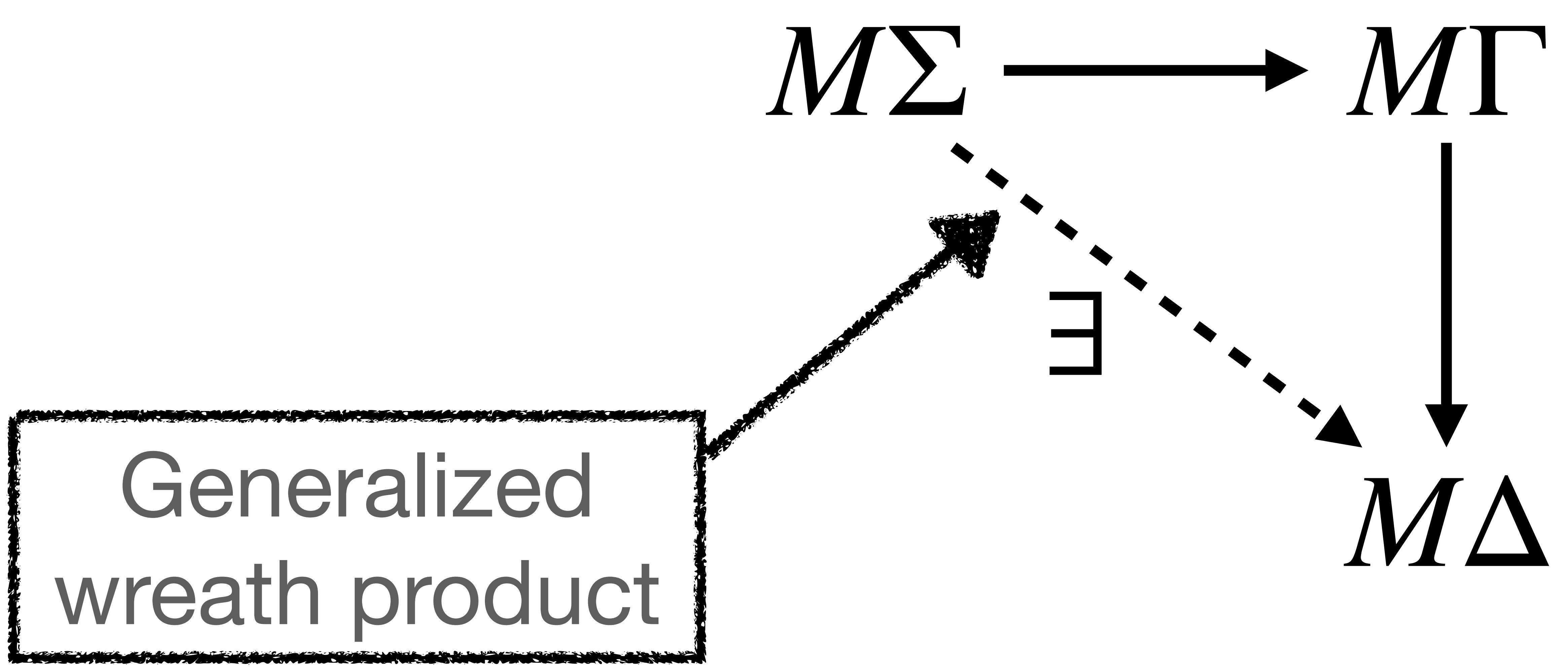

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Theorem

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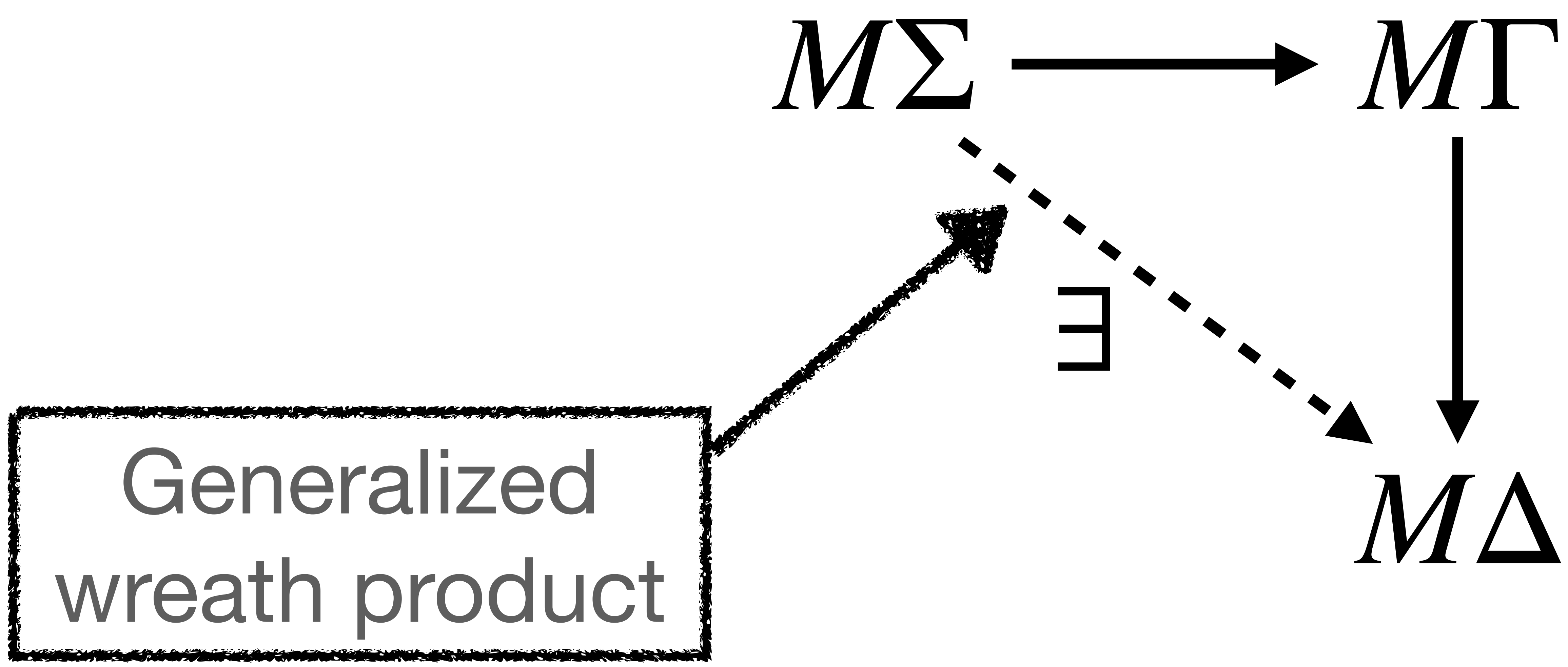

Verified in Coq

This needs some axioms about the monad-comonad interactions.

Axioms

One additional operation

$$\text{put} : MA \times A \to MA$$

One additional operation

$$\text{put} : MA \times A \rightarrow MA$$

Replace the focused element of a comonad

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$$\text{put} : MA \times A \to MA$$

Replace the focused element of a comonad

$$([1,2,3,4],\ 5) \mapsto [1,2,3,5]$$

Axioms

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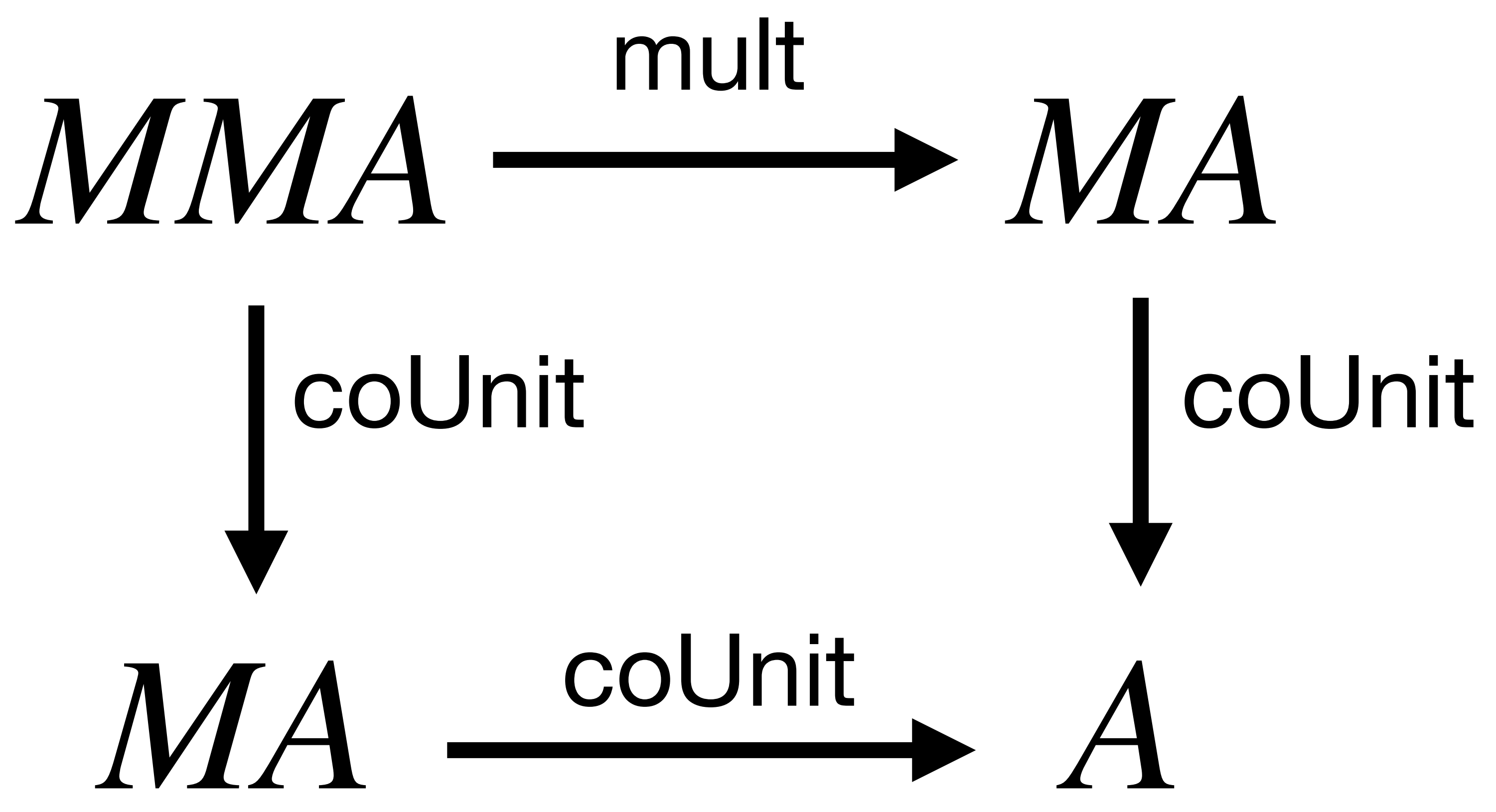

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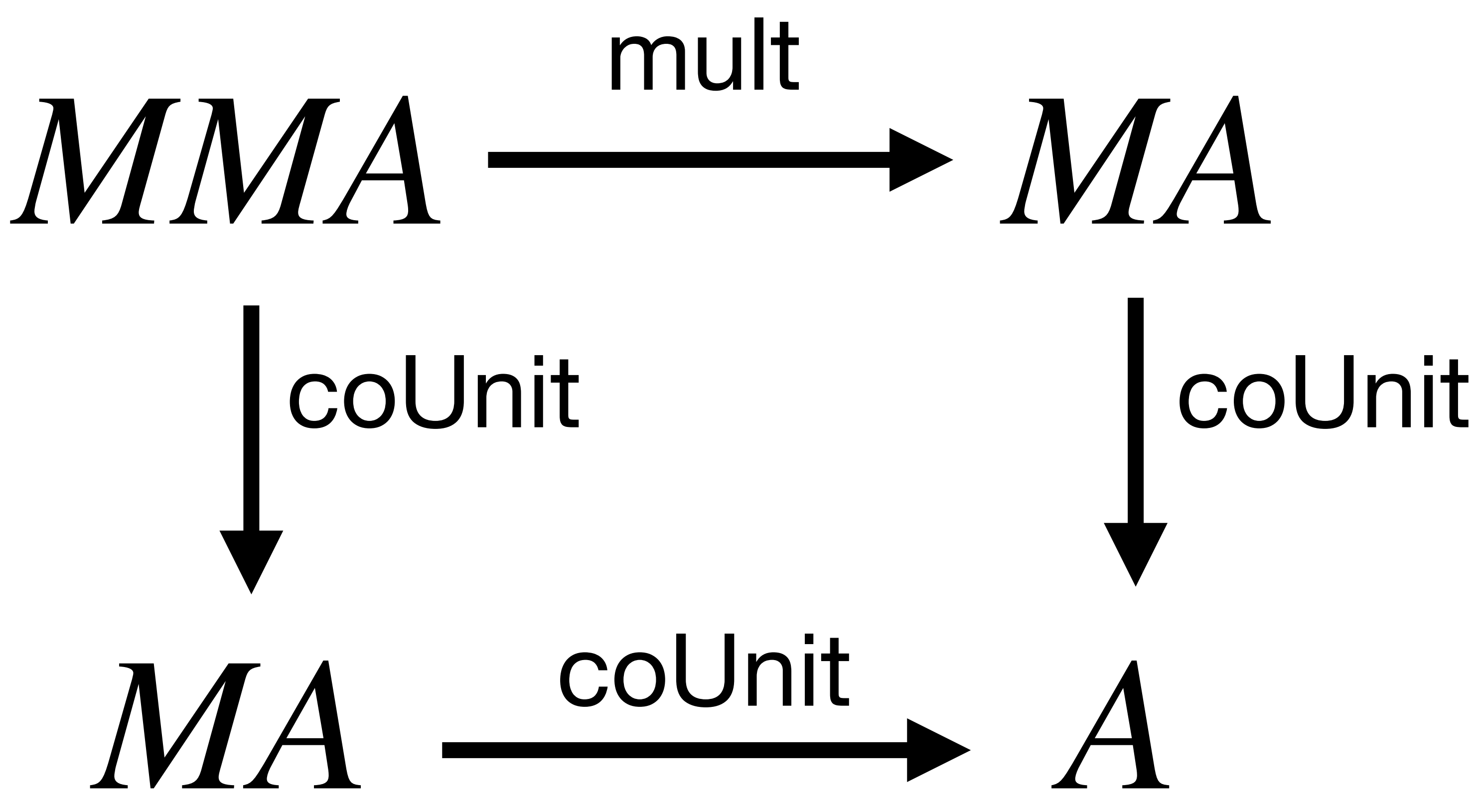

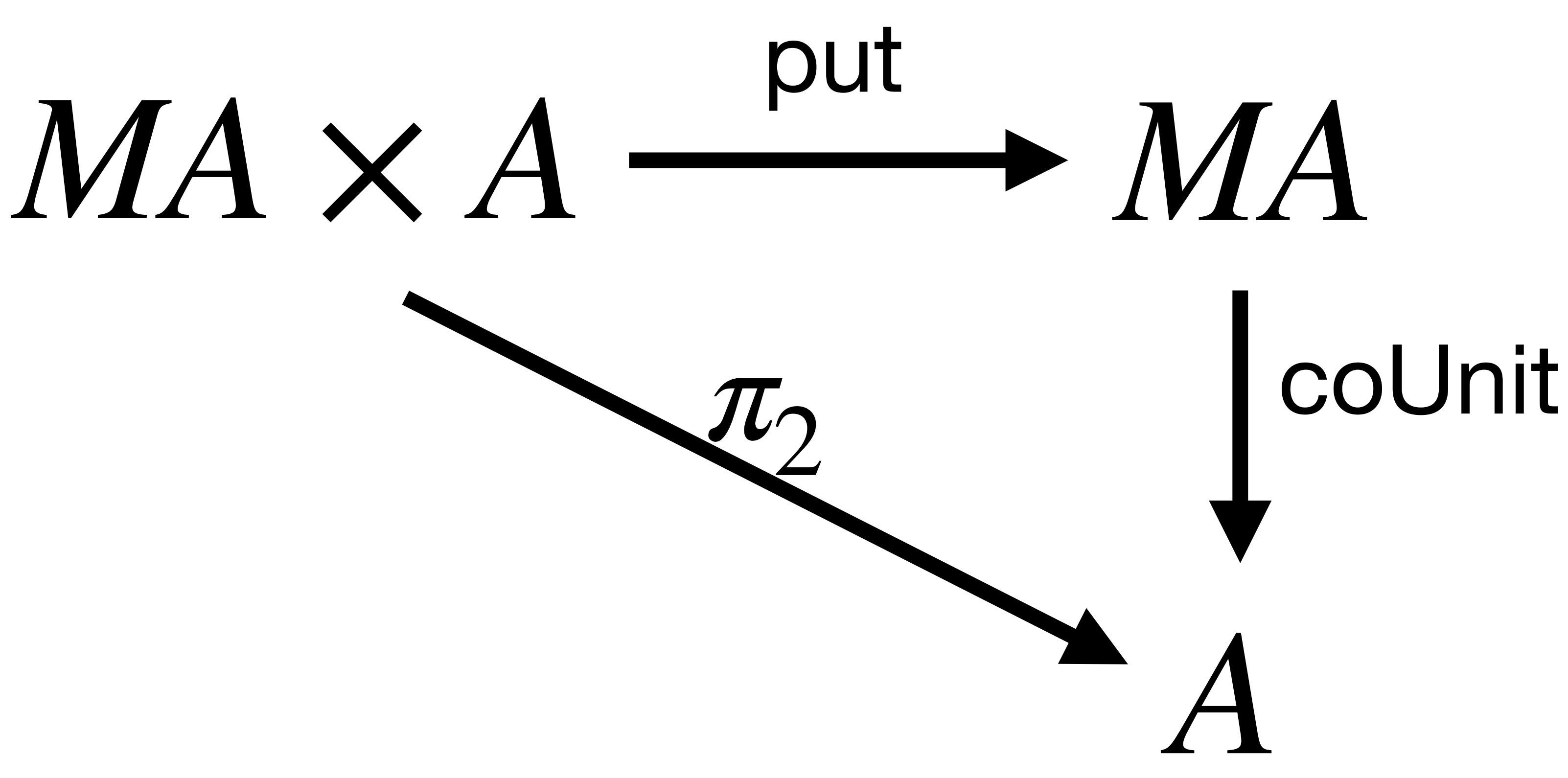

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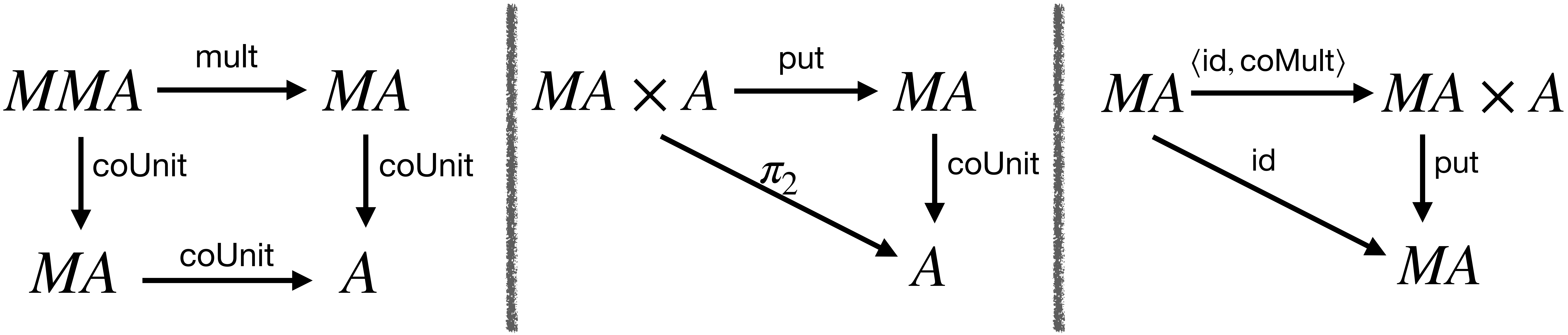
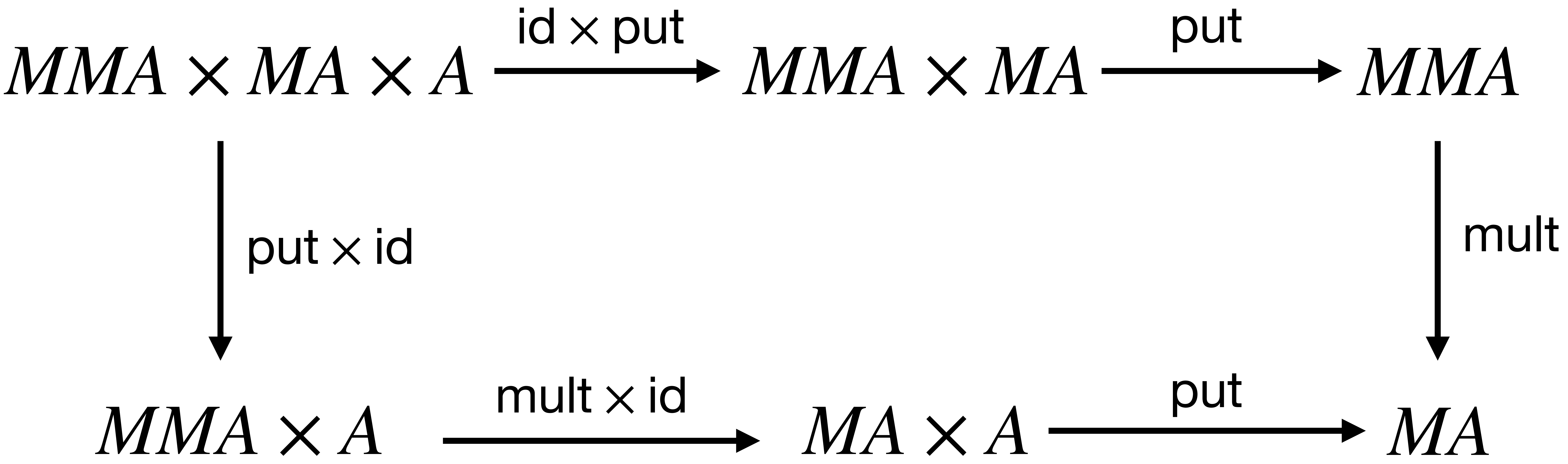

MMA
mult
MA
coUnit
coUnit
MA
coUnit
A
MA × A
put
MA
π2
coUnit
A
MA
⟨id, coMult⟩
MA × A
id
put
MA

Axioms



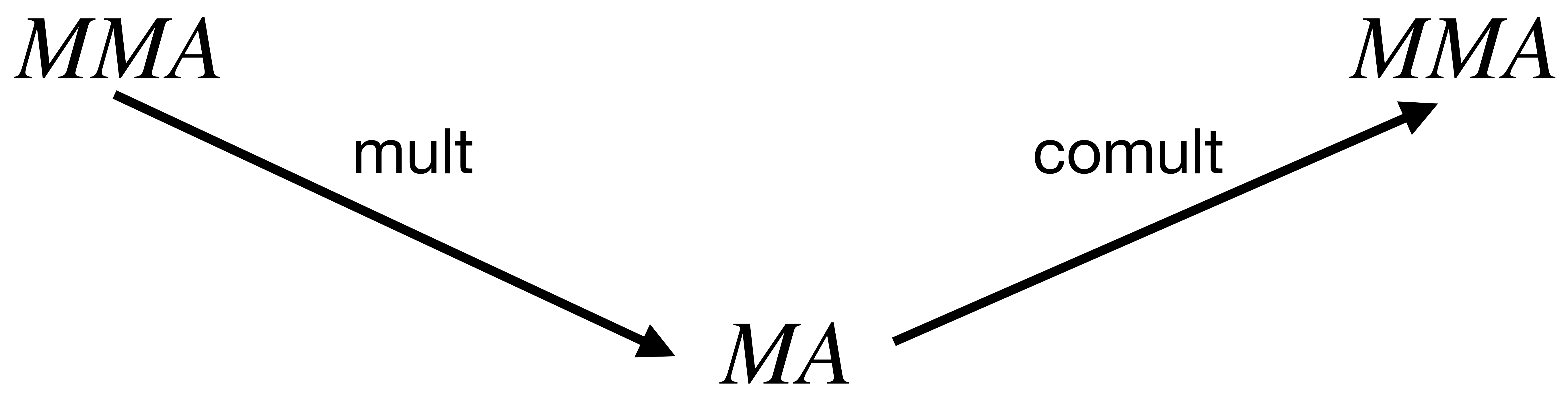
$MMA \times MA \times A$
id × put
$MMA \times MA$
put
MMA
put × id
mult
$MMA \times A$
mult × id
$MA \times A$
put
MA

Axioms

$$
\begin{array}{ccccc}
MMA \times MA \times A & \xrightarrow{\text{id} \times \text{put}} & MMA \times MA & \xrightarrow{\text{put}} & MMA \\
\Big\downarrow{\scriptstyle \text{put} \times \text{id}} & & & & \Big\downarrow{\scriptstyle \text{mult}} \\
MMA \times A & \xrightarrow{\text{mult} \times \text{id}} & MA \times A & \xrightarrow{\text{put}} & MA
\end{array}
$$

Associativity of put

Axioms



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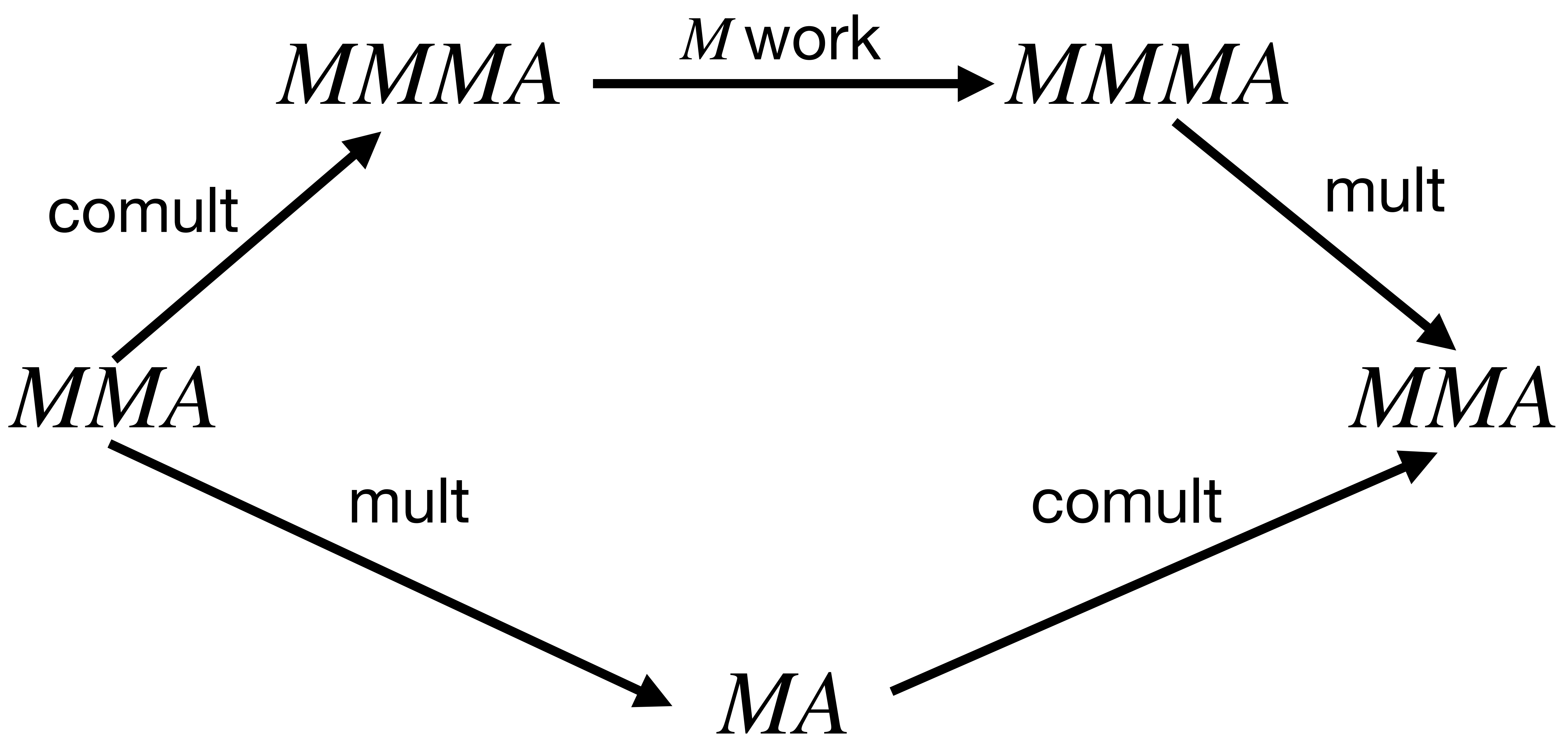

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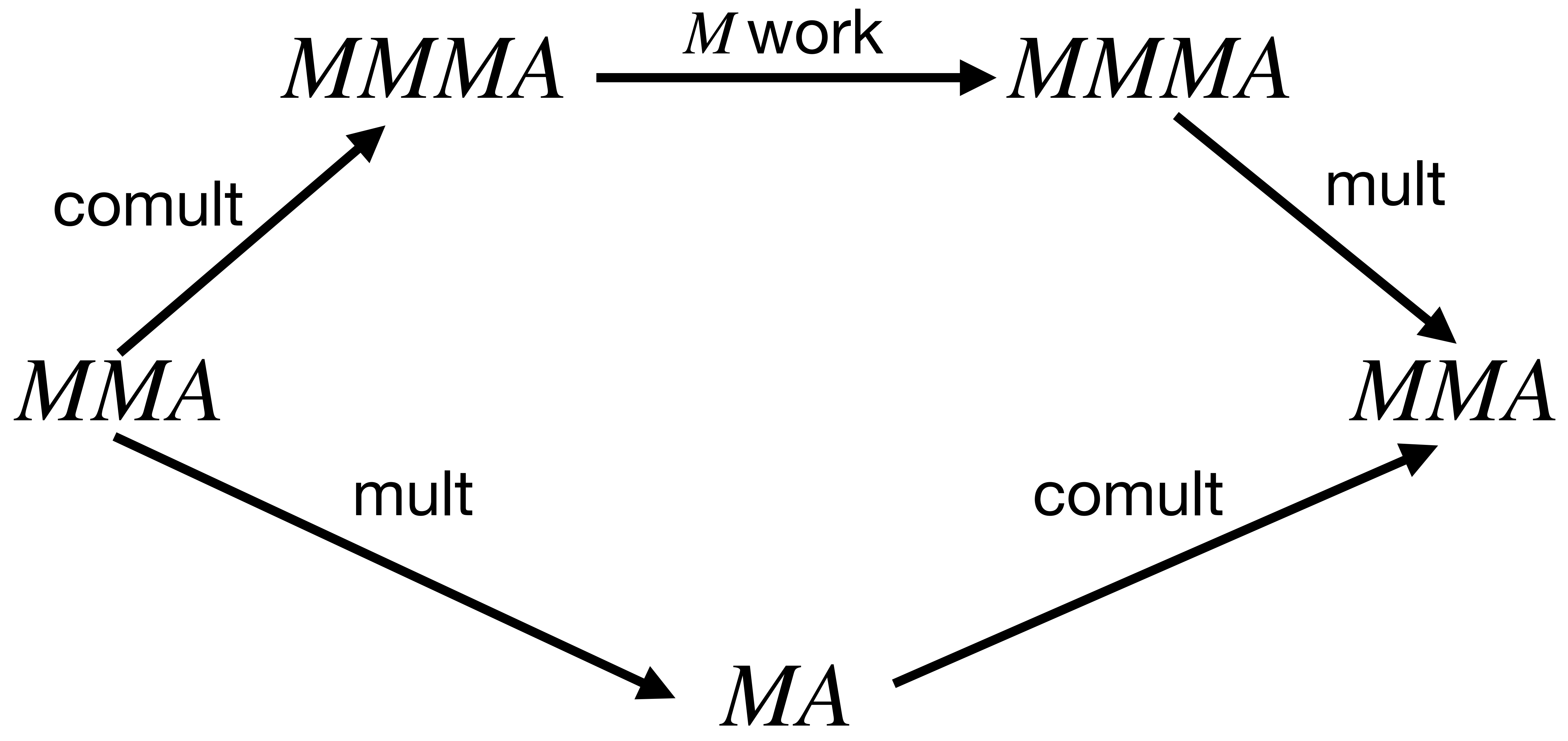

Where work is defined as follows:

$$MMA \xrightarrow{\langle \mathrm{id}, \mathrm{coUnit} \rangle} MMA \times MA \xrightarrow{\mathrm{id} \times \mathrm{coMult}} MMA \times MMA \xrightarrow{\mathrm{strength}} M(MMA \times MA) \xrightarrow{M\,\mathrm{put}} MMMA \xrightarrow{M\,\mathrm{mult}} MMA$$

Example

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$$[[1,2,3], [4,5,6], [7,8,9]]$$

Example

[[1,2,3], [4,5,6], [7,8,9]]

↓ comult

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[[1,2,3], [4,5,6], [7,8,9]]

↓ comult

[[[1,2,3]], [[1,2,3], [4,5,6]], [[1,2,3], [4,5,6], [7,8,9]]]

Example

$$[[1,2,3],[4,5,6],[7,8,9]]$$

$$\downarrow \text{comult}$$

$$[\,[[1,2,3]],\ [[1,2,3],[4,5,6]],\ [[1,2,3],[4,5,6],[7,8,9]]\,]$$

$$\downarrow M\,\langle \text{id}, \text{coUnit} \rangle$$

Example

$$[[1,2,3],[4,5,6],[7,8,9]]$$

↓ comult

$$[\,[[1,2,3]],\ [[1,2,3],[4,5,6]],\ [[1,2,3],[4,5,6],[7,8,9]]\,]$$

↓ $M\,\langle \mathsf{id}, \mathsf{coUnit}\rangle$

$$[\,([[1,2,3]],\ [1,2,3]),\ ([[1,2,3],[4,5,6]],\ [4,5,6]),\ ([[1,2,3],[4,5,6],[7,8,9]],\ [7,8,9])\,]$$

Example

$$[[1,2,3], [4,5,6], [7,8,9]]$$

$\downarrow$ comult

$$[\ [[1,2,3]],\ [[1,2,3], [4,5,6]],\ [[1,2,3], [4,5,6], [7,8,9]]\ ]$$

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$\downarrow$ $M\,(\text{id} \times \text{coMult})$

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Example (continued)

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Example (continued)

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↓ M strength

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↓ MM put

Example (continued)

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↓ MM put

[…, [[[1,2,3], [4]], [[1,2,3], [4,5]], [[1,2,3], [4,5,6]]], …]

↓ MM mult

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↓ MM put

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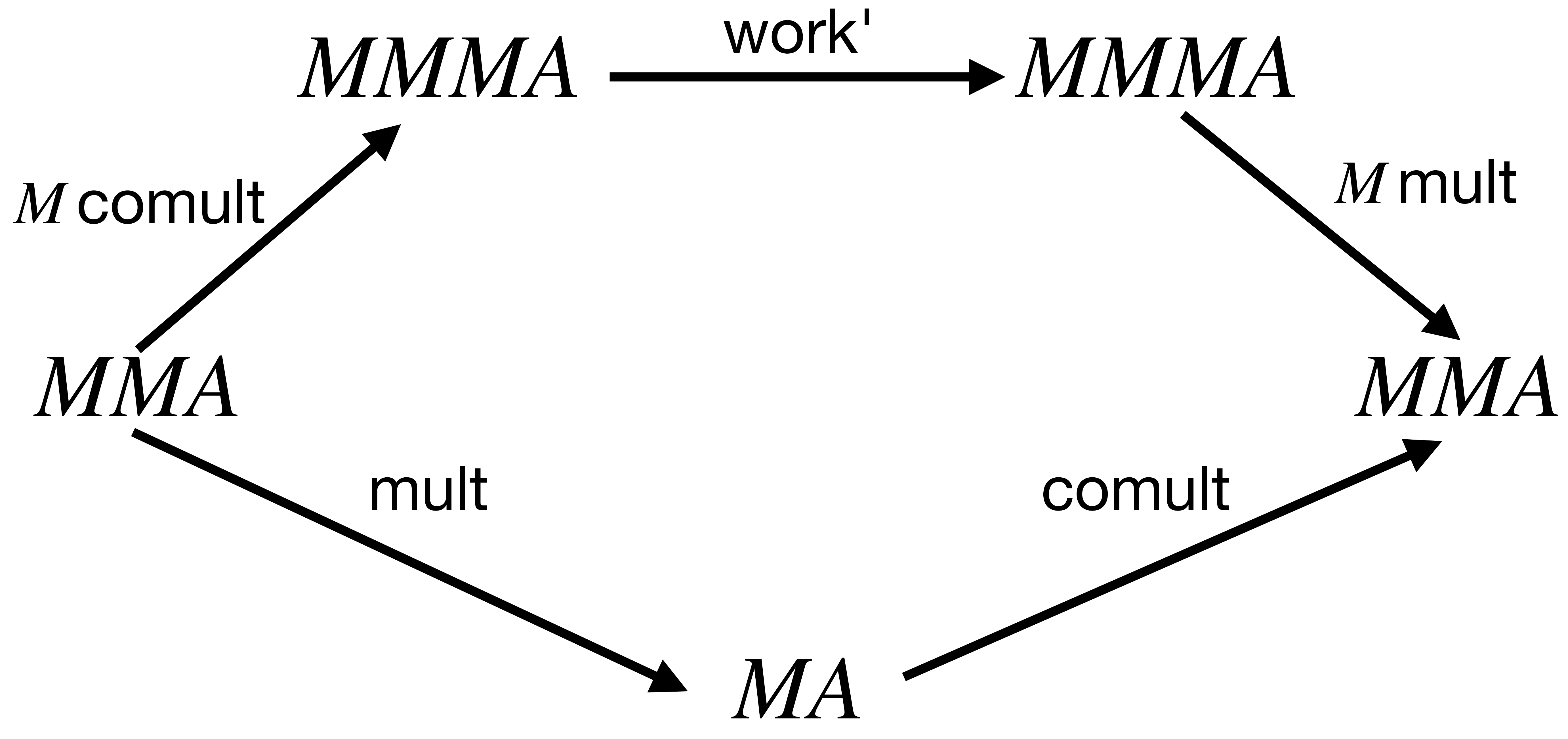
↓ MM mult

[…, [[1,2,3,4], [1,2,3,4,5], [1,2,3,4,5,6]], …]

Alternative formulation



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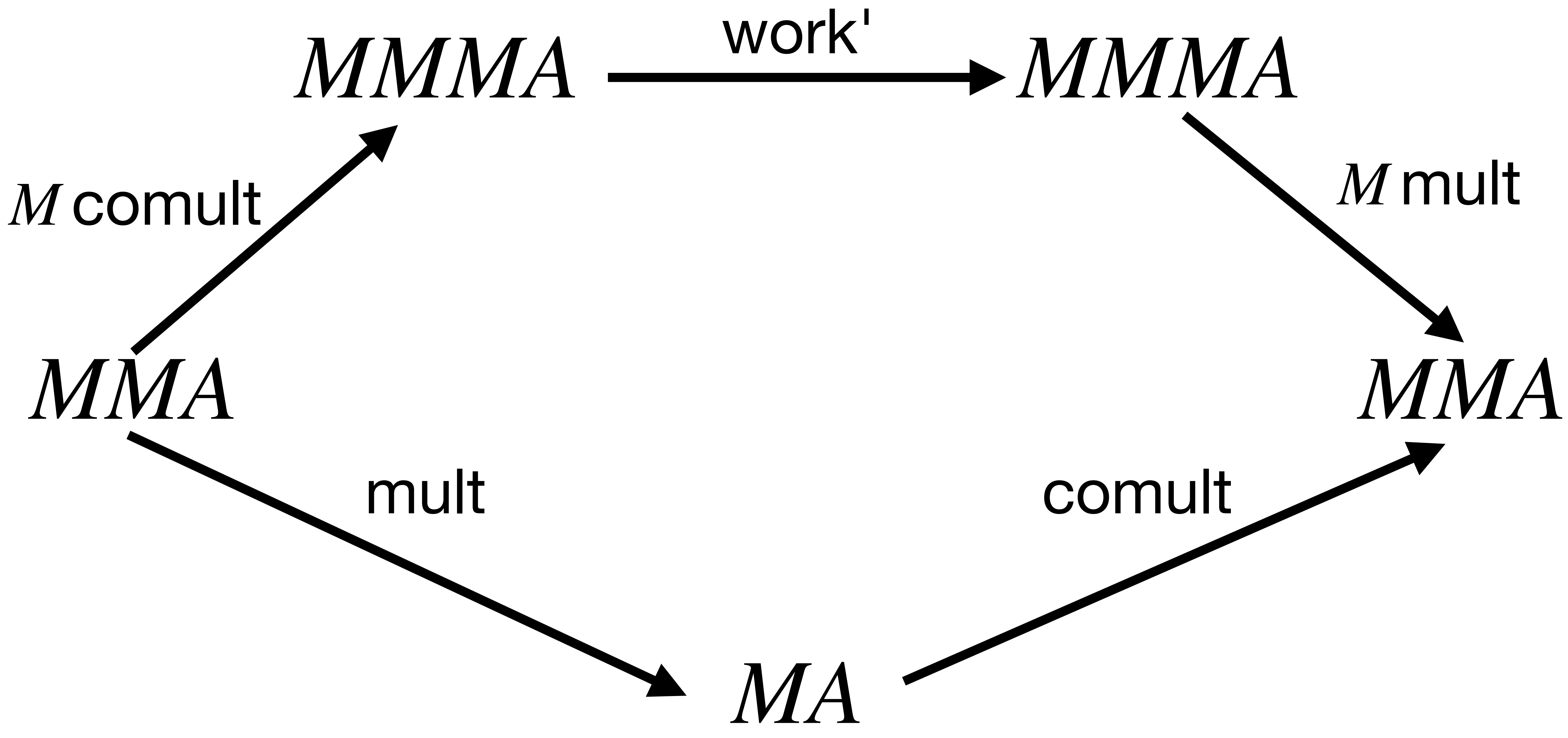

Bialgebra?

Other categories?

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Problems with axiomatization of strength:

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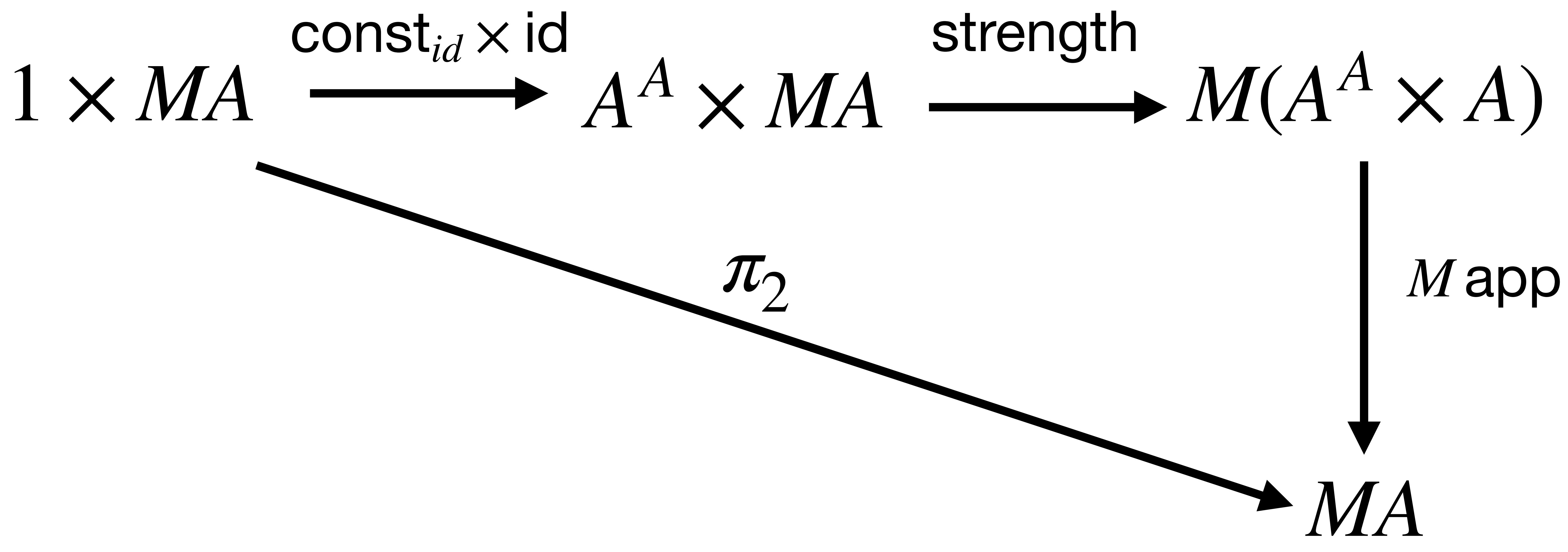
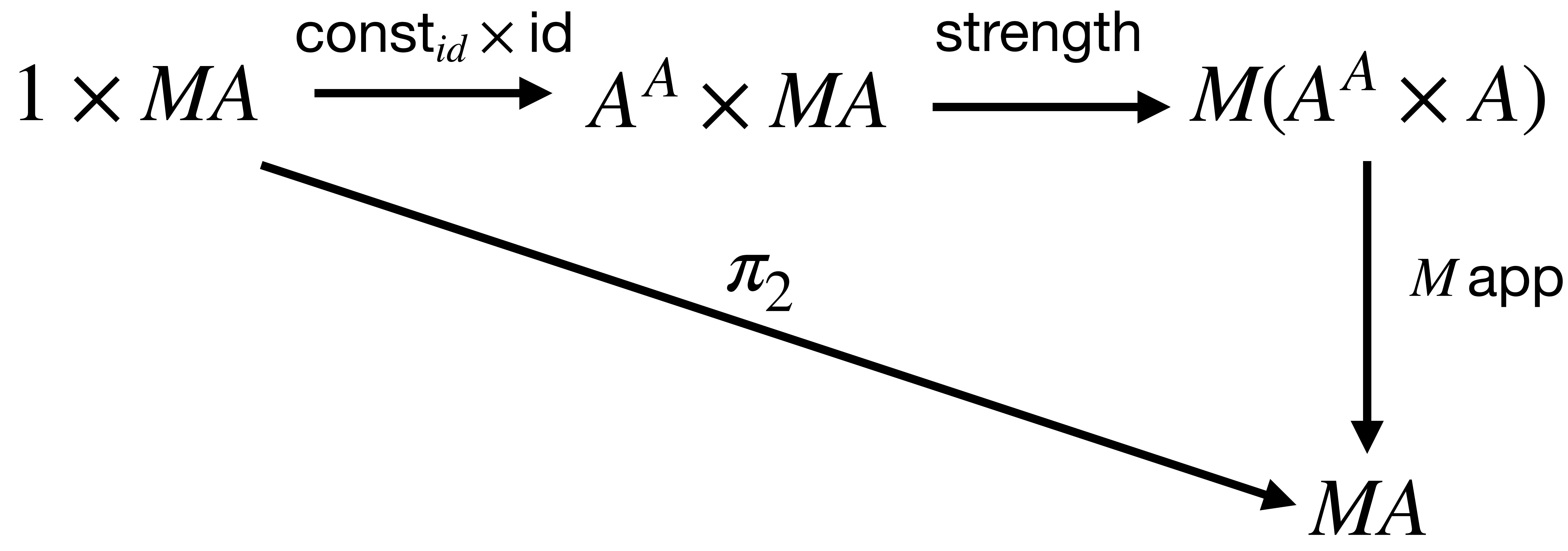


Other categories?

Problems with axiomatization of strength:



Thank you!