Dimension theory for families of sets

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Forging new atoms

- $\begin{pmatrix} \forall \vec{x} & \exists y \\ \forall \vec{u} & \exists v \end{pmatrix} \phi \;\equiv\; \forall \vec{x} \forall \vec{u} \exists y \exists v (=\!\!(\vec{x}, y) \wedge =\!\!(\vec{u}, v) \wedge \phi)$
- Old atoms: $x = y$, $R(x_1, \ldots, x_n)$
- New atoms: $=\!\!(x_1, \ldots, x_n, y)$, $=\!\!(x)$ (V. 2007)
- From individual assignments to sets of assignments.
- Truth values are families of sets of assignments, not sets of assignments.
- "From IF to BI" (Abramsky-V. 2009).
- Intuitionistic implication $\phi \to \psi$: "every subfamily of type ϕ is of type ψ".
- $\models \; =\!\!(x_1, \ldots, x_n, y) \;\equiv\; (=\!\!(x_1) \wedge \ldots \wedge =\!\!(x_n)) \;\to\; =\!\!(y)$

Propositional operators on families of sets

$$\begin{cases} \Delta_{\cup}(\mathcal{A},\mathcal{B}) & = & \mathcal{A} \cup \mathcal{B} \\ \Delta_{\cap}(\mathcal{A},\mathcal{B}) & = & \mathcal{A} \cap \mathcal{B} \\ \Delta_{c}(\mathcal{A}) & = & \mathcal{P}(X) \setminus \mathcal{A} \end{cases}$$

$$\begin{cases} \Delta_{\vee}(\mathcal{A},\mathcal{B}) & = & \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \\ \Delta_{\wedge}(\mathcal{A},\mathcal{B}) & = & \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \\ \Delta_{\neg}(\mathcal{A}) & = & \{X \setminus A \mid A \in \mathcal{A}\} \end{cases}$$

$$\Delta_{\rightarrow}(\mathcal{A},\mathcal{B}) \;=\; \{C \mid \forall D \subseteq C (D \in \mathcal{A} \Rightarrow D \in \mathcal{B})\}$$

Quantifier operators on families of sets

$$\begin{aligned}
\Delta_{\exists i}(\mathcal{A}) &= \{f[A] \mid A \in \mathcal{A}\} \text{ where} \\
& f(a_0, \ldots, a_{m-1}) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m-1}) \\
\Delta_{\forall i}(\mathcal{A}) &= \{B \mid B[X/i] \in \mathcal{A}\}, \text{ where } B[X/i] = \\
& \{(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{m-1}) \mid \\
& \ (a_0, \ldots, a_{i-1}, \quad a_{i+1}, \ldots, a_{m-1}) \in B, a_i \in X\}
\end{aligned}$$

(Abramsky-V. (2009) gives a category-theoretic justification of these definitions.)

Truth values

Fix a model M.

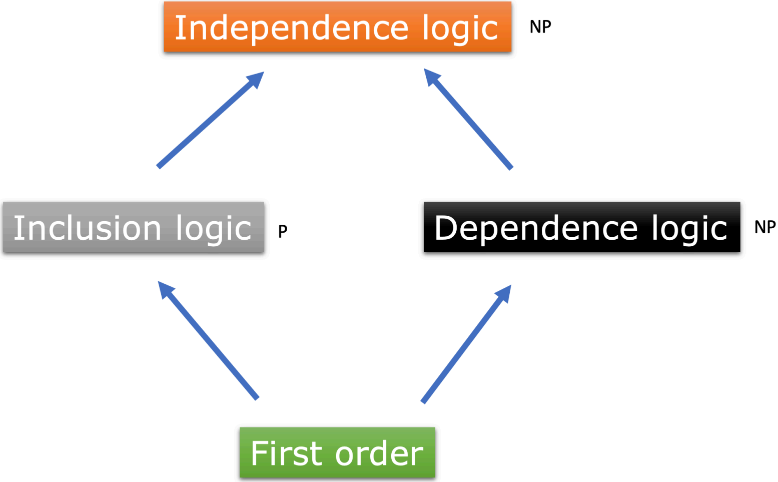
$$
\begin{aligned}
\|\phi\|^M &= \mathcal{P}(\{\vec{a} \in M^m \mid M \models \phi(\vec{a})\}), \text{ for } \phi(\vec{x}) \text{ a literal} \\
\|\phi \wedge \psi\|^M &= \Delta_{\cup}(\|\phi\|^M, \|\psi\|^M) \\
\|\phi \vee \psi\|^M &= \Delta_{\vee}(\|\phi\|^M, \|\psi\|^M) \\
\|\exists x_i \phi\|^M &= \Delta_{\exists i}^{M^m}(\|\phi\|^M) \\
\|\forall x_i \phi\|^M &= \Delta_{\forall i}^{M^m}(\|\phi\|^M),
\end{aligned}
$$

The atomic level: new atoms

- **Dependence atom**: $\|{=}(\vec{x}, y)\|^M$ is the family of sets T of assignments such that $s(\vec{x}) = s'(\vec{x})$ implies $s(y) = s'(y)$ for all $s, s' \in T$.
- **Inclusion atom**: $\|\vec{x} \subseteq \vec{y}\|^M$ is the family of sets T of assignments such that for every $s \in T$ there is $s' \in T$ such that $s(\vec{x}) = s'(\vec{y})$.
- **Independence atom**: $\|\vec{x} \perp \vec{y}\|^M$ is the family of sets T of assignments such that for every $s, s' \in T$ there is $s'' \in T$ such that $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.

New logics

New atom	New logic ($\vee, \wedge, \forall, \exists$)
$=\!\!(x, y)$	Dependence logic
$x \subseteq y$	Inclusion logic
$x \perp y$	Independence logic


Independence logic
NP
Inclusion logic
P
Dependence logic
NP
First order

Toward a dimension analysis of these and related logics

- $\mathcal{A}$ is *convex* if $\forall C(A \subseteq C \subseteq B \Rightarrow C \in \mathcal{A})$ for all $A, B \in \mathcal{A}$.
- $\mathcal{A}$ is *dominated (by $\bigcup \mathcal{A}$)* if $\bigcup \mathcal{A} \in \mathcal{A}$.
- $\mathcal{G} \subseteq \mathcal{A}$ *dominates* $\mathcal{A}$ if there exist dominated convex families $\mathcal{A}_G$, $G \in \mathcal{G}$, such that $\bigcup_{G \in \mathcal{G}} \mathcal{A}_G = \mathcal{A}$ and $\bigcup \mathcal{A}_G = G$, for each $G \in \mathcal{G}$.
- The **dimension** of $\mathcal{A}$:

$$\mathrm{D}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ dominates the family } \mathcal{A}\},$$

Dimensions of some families

Theorem

Suppose $\ell = |X| \geq 2$ and $n = |Y| \geq 2$. Then:

$$
\begin{aligned}
\mathrm{D}(\{f \subseteq X \times Y \mid f \textit{ is a mapping }\}) &= n^{\ell} \\
\mathrm{D}(\{R \subseteq X \times X \mid \mathrm{dom}(R) \subseteq \mathrm{rg}(R)\}) &= 2^{\ell} - \ell \\
\mathrm{D}(\{A \times B \mid A \subseteq X,\ B \subseteq Y\}) &= \\
&(2^{\ell} - \ell - 1)(2^{n} - n - 1) + \ell + n
\end{aligned}
$$

Dimensions of some atoms

Suppose $|M| = n$.

ϕ	$\mathrm{D}(\|\phi\|^M)$	
$x = y$	1	
$x \neq y$	1	
$R(\vec{x})$	1	
$\neg R(\vec{x})$	1	
$=\!(y)$	n	
$=\!(\vec{x}, y)$	n^{n^m}	$\mathrm{len}(\vec{x}) = m$
$\vec{x} \subseteq \vec{y}$	$2^{n^m} - n^m$	$\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = m$
$\vec{x} \perp \vec{y}$	$\approx 2^{n^m + n^k}$	$\mathrm{len}(\vec{x}) = m, \mathrm{len}(\vec{y}) = k$

Growth classes

- $\mathbb{E}_k$ is the set of $f : \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial p of degree k such that $f(n) \leq 2^{p(n)}$.
- $\mathbb{F}_k$ is the set of functions $f : \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial p of degree k such that $f(n) \leq n^{p(n)}$.
- $\mathbb{E}_0 \subsetneq \mathbb{F}_0 \subsetneq \mathbb{E}_1 \subsetneq \mathbb{F}_1 \subsetneq \cdots \subsetneq \mathbb{E}_k \subsetneq \mathbb{F}_k$.
- Note that $\mathbb{E}_0$ is the class of bounded functions and $\mathbb{F}_0$ the class of functions of polynomial growth.

The dimension of a formula

$$\boxed{\mathrm{Dim}_{\phi}(n) = \sup\left\{\mathrm{D}(\|\phi\|^{M}) \mid M \text{ is a model}, |M| = n\right\}}$$

1. $\mathrm{Dim}_{\phi,\vec{x}}(n) = 1$, hence Dim_{ϕ} is in $\mathbb{E}_0$, for every first order ϕ.
2. $\mathrm{Dim}_{=(\vec{x},y)}(n) = n^{n^k}$, hence $\mathrm{Dim}_{=(\vec{x},y)}$ is in $\mathbb{F}_k$, where $\mathrm{len}(\vec{x}) = k$.
3. $\mathrm{Dim}_{\vec{x}\subseteq\vec{y}}(n) = 2^{n^k} - n^k$, hence $\mathrm{Dim}_{\vec{x}\subseteq\vec{y}}$ is in $\mathbb{E}_k$, where $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = k$.
4. $\mathrm{Dim}_{\vec{x}\perp\vec{y}}(n) = (2^{n^m} - n^m - 1)(2^{n^k} - n^k - 1) + n^m + n^k$, hence $\mathrm{Dim}_{\vec{x}\perp\vec{y}}$ is in $\mathbb{E}_{m+k}$, where $\mathrm{len}(\vec{x}) = k$ and $\mathrm{len}(\vec{y}) = m$.

Theorem

Let $\mathbb{O}$ be a growth class (i.e. some $\mathbb{E}_i$ or $\mathbb{F}_i$). Furthermore, let $\phi = \phi(\vec{x})$ and $\psi = \psi(\vec{x})$ be formulas of some logic $\mathcal{L}$ with team semantics.

(a) *If ϕ is a literal, then* $\mathrm{Dim}_\phi \in \mathbb{O}$.

(b) *If* $\mathrm{Dim}_\phi, \mathrm{Dim}_\psi \in \mathbb{O}$, *then* $\mathrm{Dim}_{\phi \wedge \psi} \in \mathbb{O}$.

(c) *If* $\mathrm{Dim}_\phi, \mathrm{Dim}_\psi \in \mathbb{O}$, *then* $\mathrm{Dim}_{\phi \vee \psi} \in \mathbb{O}$.

(d) *If* $\mathrm{Dim}_\phi \in \mathbb{O}$, *then* $\mathrm{Dim}_{\exists x_i \phi} \in \mathbb{O}$ *and* $\mathrm{Dim}_{\forall x_i \phi} \in \mathbb{O}$.

How is the theorem proved?

Definition

Let X and Y be nonempty sets. A function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is a Kripke-operator, if there is a relation $\mathcal{R} \subseteq \mathcal{P}(Y) \times \mathcal{P}(X)^n$ such that

$$B \in \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) \iff$$
$$\exists A_0 \in \mathcal{A}_0 \ldots \exists A_{n-1} \in \mathcal{A}_{n-1} : \ (B, A_0, \ldots, A_{n-1}) \in \mathcal{R}.$$

- $\Delta_\cap$ is a Kripke-operator.[1]
- $\Delta_\vee$ and $\Delta_\neg$ on X are Kripke-operators.[2]
- $\Delta_{\exists i}$ and $\Delta_{\forall i}$ are Kripke-operators.
- $\Delta_\cup$ is **not** a Kripke-operator.
- Δ_c is **not** a Kripke-operator
- $\Delta_\rightarrow$ is **not** a Kripke-operator

[1] If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ and $C \in \mathcal{P}(X)$, then $C \in \mathcal{A} \cap \mathcal{B}$ if and only if there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $(C, A, B) \in \mathcal{R}_\cap$, where $\mathcal{R}_\cap$ is the relation $\{(D, D, D) \mid D \in \mathcal{P}(X)\}$.

[2] $\mathcal{A} \vee \mathcal{B} = \Delta_{\mathcal{R}_\vee}(\mathcal{A}, \mathcal{B})$ and $\Delta_\neg^X(\mathcal{A}) = \Delta_{\mathcal{R}_\neg}(\mathcal{A})$ where $\mathcal{R}_\vee = \{(A \cup B, A, B) \mid A, B \in \mathcal{P}(X)\}$ and $\mathcal{R}_\neg = \{(X \setminus A, A) \mid A \in \mathcal{P}(X)\}$.

Definition

We say that Δ *weakly preserves dominated convexity* if $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$ is dominated and convex (or $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) = \emptyset$) whenever $\mathcal{A}_i$ is dominated and convex for each $i < n$.

Theorem

Let $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ be a Kripke-operator, and let $\mathcal{A} = \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$. If Δ weakly preserves dominated convexity then

$$\mathrm{D}(\Delta_{\mathcal{R}}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})) \leq \mathrm{D}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}(\mathcal{A}_{n-1})$$

.

Below we will use the notation

$$\mathcal{R}[A] := \{(A_0, \ldots, A_{n-1}) \mid (A, A_0, \ldots, A_{n-1}) \in \mathcal{R}\}.$$

Definition (Lück 2020)

A Kripke-operator $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is *local* if, for any $A \in \mathcal{P}(Y)$, $\mathcal{R}[A]$ is determined by the relations $\mathcal{R}[\{a\}]$, $a \in A$, as follows:

$(A_0, \ldots, A_{n-1}) \in \mathcal{R}[A] \iff$ *for each* $a \in A$ *there is* $(A_0^a, \ldots, A_{n-1}^a) \in \mathcal{R}[\{a\}]$ *such that* $A_i = \bigcup_{a \in A} A_i^a$ *for* $i < n$.

Theorem

If $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ *is a local Kripke-operator, then it weakly preserves dominated convexity.*

Theorem

The operators $\Delta_{\cap}^{M^m}$, $\Delta_{\vee}^{M^m}$, $\Delta_{\exists i}^{M^m}$ *and* $\Delta_{\forall i}^{M^m}$ *are local.*

Hence they preserve dimension!

Definition

The logic $\mathbb{LE}_k$ is the closure of literals and all atoms whose dimension function is in the growth class $\mathbb{E}_k$ under the connectives $\wedge$, $\vee$ and any Lindström quantifiers. Similarly $\mathbb{LF}_k$ for $\mathbb{F}_k$.

Lemma

(a) $\mathbb{LE}_k \subseteq \mathbb{LF}_k \subseteq \mathbb{LE}_{k+1} \subseteq \mathbb{LF}_{k+1}$.

Note:

(a) The dimension of every formula in $\mathbb{LE}_k$ is in the growth class $\mathbb{E}_k$.

(b) The dimension of every formula in $\mathbb{LF}_k$ is in the growth class $\mathbb{F}_k$.

The arity-concept

Definition

- The atom $=\!(\vec{x}, y)$ is k-ary, if $\text{len}(\vec{x}) = k$,
- The atom $\vec{x} \subseteq \vec{y}$ is k-ary if $\text{len}(\vec{x}) = \text{len}(\vec{y}) = k$,
- The atom $\vec{t_2} \perp \vec{t_3}$ is $\max(k, l)$-ary, if $\text{len}(\vec{t_2}) = k$, and $\text{len}(\vec{t_3}) = l$.

Theorem

1. *k-ary inclusion and independence logics are included in $\mathbb{LE}_k$.*
2. *The k-ary dependence logic is included in $\mathbb{LF}_k$.*
3. *The (k, l)-ary independence logic is included in $\mathbb{LF}_{\max(k,l)}$.*

Theorem

1. *The $k+1$-ary inclusion, anonymity, exclusion and independence atoms are not definable in $\mathbb{LE}_k$.*
2. *The $k+1$-ary dependence atom is not definable in $\mathbb{LF}_k$.*
3. *The (k, l)-ary independence atom is not definable in $\mathbb{LF}_i$ if $i < \max(k, l)$.*

Hence:

Dependence logic, inclusion logic, and pure independence logic each has a proper definability hierarchy for formulas based on the arity of the non-first order atoms.

But the above result is, of course, much stronger.

An application to intuitionistic implication

$$=\!\!(x_1, \ldots, x_n, y) \;\equiv\; (=\!\!(x_1) \wedge \ldots \wedge =\!\!(x_n)) \;\color{red}{\rightarrow}\; =\!\!(y)$$

exponential *linear* *linear*

Ergo: $\rightarrow$ is exponential[3]

[3] and not definable from dependence, inclusion, or independence atoms even if Lindström quantifiers are added.

An application to intuitionistic disjunction

$$\|\phi \veebar \psi\|^M = \|\phi\|^M \cup \|\psi\|^M$$

$x = y \veebar x \neq y$ has dimension 2

Ergo: $\veebar$ cannot be defined in first order logic.

$$\mathcal{M} \models_X \phi \odot \psi \iff$$

$$\forall_{\neq\emptyset} Y, Z \subseteq X((\mathcal{M} \models_Y \phi \text{ and } \mathcal{M} \models_Z \psi) \to$$
$$\exists Y', Z' \subseteq X(Y \subseteq Y', Z \subseteq Z', \mathcal{M} \models_{Y'} \phi, \mathcal{M} \models_{Z'} \psi,$$
$$\text{and } Y' \cap Z' \neq \emptyset)).$$

$$x \perp y \iff =\!(x) \odot =\!(y)$$

Ergo: $\odot$ is exponential and not definable from dependence and inclusion atoms, even if Lindström quantifiers are added.

Many open problems:

1. Is the k-ary dependence atom definable in the extension of first order logic by k-ary independence, exclusion, inclusion, anonymity, constancy atoms, and some Lindström quantifiers?
2. Is the k-ary anonymity atom definable in terms of the k-ary inclusion atom?
3. Is the (k, l, m)-ary independence atom definable in terms of the $\max(k, l) + m$-ary dependence atom, $\max(k, l) + m$-ary, $\max(k, l) + m$-ary exclusion atoms, and the $\max(k, l) + m$-ary inclusion atom?

Note: Sentences have dimension 1, so dimension theory cannot be used to obtain hierarchy results for sentences.

Congratulations Samson

for the incredible book,

and many happy returns!