GoI to SmP
Samson at UCL, 19 September 2023

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In the early nineties Samson and Radha Jagadeesan provided new foundations for GoI (Girard's Geometry of Interaction).

That idea returns in understanding two-sided games and strategies over relational structures within the programme SmP (Structure meets Power), which began in the relatively recent work of Samson, Anuj Dawar and Pengming Wang in providing unity to arguments in Finite Model Theory.

Their central idea: strategies in one-sided Spoiler-Duplicator games are coKleisli maps w.r.t. a comonad over homomorphisms between structures.
Composition of strategies = composition of coKleisli maps
— not obviously the usual composition of strategies!

Thanks to: Adam Ó Conghaile, Mai Gehrke, Sacha Huriot-Tattegrain, Yoav Montacute

The "usual" composition of strategies

In 2-party games read Player vs. Opponent as Process vs. Environment.
Follow the paradigm of Conway, Joyal to achieve compositionality.

Assume operations on (2-party) games:

Dual game $A^\perp$ *- interchange the role of Player and Opponent;*
Counter-strategy = strategy for Opponent = strategy for Player in dual game.

Parallel composition of games $A \| B$.

A strategy (for Player) from a game A to a game B = strategy in $A^\perp \| B$.
A strategy (for Player) from a game B to a game C = strategy in $B^\perp \| C$.

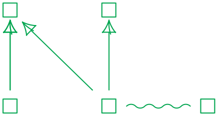
Compose by letting them play against each other in the common game B.

$\rightsquigarrow$ *a (bi)category with identity w.r.t. composition, the Copycat strategy in* $A^\perp \| A$, *so from A to A ...*

Concurrent games: Conway-Joyal on event structures

An event structure *comprises* $(E, \leqslant, \#)$, *consisting of a set of* events E
- partially ordered by $\leqslant$, *the* causal dependency relation, *and*
- a binary irreflexive symmetric relation, the conflict relation,
which satisfy $\{e' \mid e' \leqslant e\}$ *is finite and* $e_1' \geqslant e_1 \# e_2 \leqslant e_2' \implies e_1' \# e_2'$.

Two events are concurrent *when neither in conflict nor causally related.*



The configurations, $\mathcal{C}(E)$, *of an event structure* E *consist of those subsets* $x \subseteq E$ *which are* Consistent: *don't have* $e \# e'$ *for any events* $e, e' \in x$, *and*
Down-closed: $e' \leqslant e \in x \implies e' \in x$.

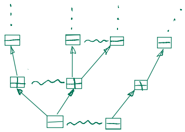
A (total) map *of event structures* $f : E \to E'$ *is a function* $f : E \to E'$ *such that*

$$\forall x \in \mathcal{C}(E).\ \ f\,x \in \mathcal{C}(E') \text{ and } e_1, e_2 \in x \ \&\ f(e_1) = f(e_2) \implies e_1 = e_2 .$$

Maps preserve concurrency and reflect causal dependency locally.

Concurrent games: Conway-Joyal on event structures

Concurrent games are represented by event structures in which events are labelled + (Player) or − (Opponent). Support dual $(\_)^{\perp}$ and parallel compn $\|$.



A concurrent strategy from A to B: A map $\sigma : S \to A^{\perp} \| B$ of ev. structures for which the copycat strategy is identity w.r.t. composition of strategies, iff [Rideau,W] σ is receptive to Opponent moves of $A^{\perp} \| B$, i.e. $\sigma x \subseteq^{-} y \Rightarrow \exists! x'.\ x \subseteq x'\ \&\ \sigma x' = y$, and only introduces new immediate dependencies $\boxminus \rightarrowtriangle_S \boxplus$.

A strategy $\sigma : S \to A^{\perp} \| B$ is deterministic when all immediate conflict in S is due to Opponent, i.e. has the form $\boxminus \sim\sim\sim \boxminus$.

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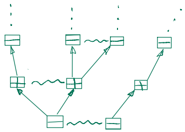

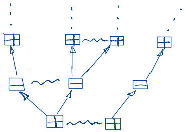

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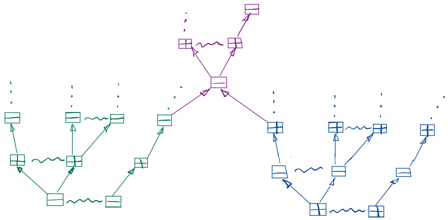

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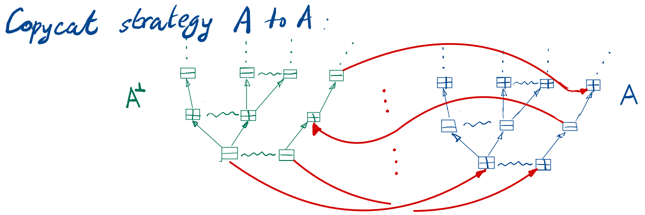

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Games supporting instantiations in Σ-structures

A signature (Σ, C, V) comprises Σ a many-sorted relational signature including equality; a set C event-name constants; a set $V = \{\alpha, \beta, \gamma, \cdots\}$ of variables.

A (Σ, C, V)-signature game comprises an event structure $(E, \leqslant, \#)$ – its moves are the events E, with

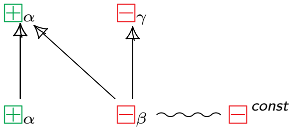
a polarity function $\mathrm{pol} : E \to \{+, -\}$ s.t. no immediate conflict $\boxplus \sim\sim\sim \boxminus$

a variable/constant assignment $\mathrm{var} : E \to C \cup V$ respecting polarity s.t.

$$e \text{ co } e' \Rightarrow \mathrm{var}(e) \neq \mathrm{var}(e')$$

a winning condition WC, an assertion in the free logic over (Σ, C, V).

E.g. WC might be
$\mathbb{E}(\gamma) \to \exists\beta.\ P(\alpha, \beta) \wedge Q(\beta)$
Existence predicate invokes only latest occurrences in a configuration

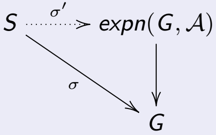


A good reference for free logic: Dana Scott, Identity and Existence. LNM 753, 1979

Games and strategies over a structure

A **game over a structure** $(G, \mathcal{A})$ *is a* (Σ, C, V)*-game* G *and* Σ*-structure* $\mathcal{A}$*.*
It determines a (traditional) concurrent game, its **expansion** $expn(G, \mathcal{A})$*, in which each* V*-move* $\square_\alpha$ *is expanded to its instances* $\square_\alpha^{a_1} \sim\sim \square_\alpha^{a_2} \sim\sim \cdots$

A **strategy** (σ, ρ) *in* $(G, \mathcal{A})$ *assigns values in* $\mathcal{A}$ *to Player moves of the game* G *in answer to assignments of Opponent. Described as a map of event structures, it corresponds to a (traditional) concurrent strategy* σ' *in* $expn(G, \mathcal{A})$*:*



For a configuration x *of* S *and a* Σ*-assertion* φ*,*
$x \models \varphi$ *will mean latest assignments to variables in* x *make* φ *true.*
The strategy is **winning** *means* $x \models WC$ *for all* $+$*-maximal configs* x *of* S*.*

Proposition. The events S of a strategy form a Σ-structure:
$R_S(s_1, \cdots, s_n)$ iff $x \models R(\rho(s_1), \cdots, \rho(s_n))$, for latest $s_1, \cdots, s_n \in x \in \mathcal{C}(S)$.
Corollary. $(G, \mathcal{A})$ determines a Σ-structure, on V-moves $expn(G, \mathcal{A})_V$.
It extends to a comonad over Σ-structures.
Event strs. provide the interaction shapes with which to build comonads!

Constructions on signature games

Let G be a (Σ, C, V)-game. Its dual $G^\perp$ is the (Σ, C, V)-game obtained by reversing polarities, i.e. the roles of Player and Opponent, with winning condition $\neg WC_G$.

Let G be a (Σ_G, C_G, V_G)-game. Let H be a (Σ_H, C_H, V_H)-game. Their parallel composition $G \| H$ is the $(\Sigma_G + \Sigma_H, C_G + C_H, V_G + V_H)$-game comprising the parallel juxtaposition of event structures with winning condition $WC_G \vee WC_H$.

Let $(G, \mathcal{A})$ to $(H, \mathcal{B})$ be games over structures. A winning strategy from $(G, \mathcal{A})$ to $(H, \mathcal{B})$ comprises a winning strategy in the game $(G^\perp \| H, \mathcal{A} + \mathcal{B})$ — its winning condition is $WC_G \to WC_H$.

Theorem. Obtain a (bi)category of winning strategies between games over structures: winning strategies compose with the copycat strategy as identity.

Strategies as reductions: a winning strategy σ from $(G, \mathcal{A})$ to $(H, \mathcal{B})$ reduces the problem of finding a winning strategy in $(H, \mathcal{B})$ to finding a winning strategy in $(G, \mathcal{A})$. A winning strategy in $(G, \mathcal{A})$ is a winning strategy from $(\emptyset, \emptyset)$ to $(G, \mathcal{A})$; its composition with σ is a winning strategy in $(H, \mathcal{B})$.

Spoiler-Duplicator games deconstructed

A Spoiler-Duplicator game is specified by a deterministic concurrent strategy

$$\begin{array}{c} D \\ \big\downarrow \delta \\ G^{\perp} \| G \end{array}$$

which is an idempotent comonad δ in the bicategory of signature games.
Idea: D, itself a signature game, specifies the pattern of strategies from $(G, \mathcal{A})$ to $(G, \mathcal{B})$, whether they follow copycat, are all-in-one, ...

The Spoiler-Duplicator category SD_δ has maps

$$(\sigma, \rho) : \mathcal{A} \longrightarrow\!\!\!\!\!+\!\!\!> _\delta \mathcal{B}$$

those deterministic strategies (σ, ρ) from $(G, \mathcal{A})$ to $(G, \mathcal{B})$ which factor openly through δ, i.e. so

$$\begin{array}{ccc} S & \overset{open}{\dashrightarrow} & D \\ & {\scriptstyle\sigma}\searrow & \big\downarrow \delta \\ & & G^{\perp} \| G . \end{array}$$

(open = "bisimulation map")

Characterising SD_δ (for $\delta : D \to G^\perp \| G$)

Assume G has signature (Σ, V, C). For Σ-structures $\mathcal{A}$ and $\mathcal{B}$, define the partial expansion $\mathrm{expn}^-(D, \mathcal{A}+\mathcal{B})$ w.r.t. just Opponent moves. Define $D(\mathcal{A}, \mathcal{B})$ to be the set of its Player V-moves.

Strategies $\mathcal{A} \rightarrow\!\!\!\!+\!\!\!\!\!\!> _\delta \mathcal{B}$ in SD_δ correspond to sort-respecting functions

$$h : D(\mathcal{A}, \mathcal{B}) \to \mathcal{A} + \mathcal{B}$$

assigning elements of $\mathcal{A}$ and $\mathcal{B}$ to V-moves of Player. Composition à la GoI.

Assume G is one-sided, *i.e.* all its V-moves are of Player. Then,

$$h : D(\mathcal{A}) \to \mathcal{B}\,.$$

It has a coextension $h^\dagger : D(\mathcal{A}) \to D(\mathcal{B})$ (relies on the idempotence of δ).

Strategies $\mathcal{A} \rightarrow\!\!\!\!+\!\!\!\!\!\!> _\delta \mathcal{B}$ in SD_δ correspond to $h : D(\mathcal{A}) \to \mathcal{B}$ which preserve winning conditions W_G across $+$-maximal configurations of D; they compose via coextension. All the SmP coKleisli categories I know are instances.

Relation with arboreal categories?

Characterising SD_δ (for $\delta : D \to G^\perp \| G$)

Assume G has signature (Σ, V, C). For Σ-structures $\mathcal{A}$ and $\mathcal{B}$, define the partial expansion $expn^-(D, \mathcal{A} + \mathcal{B})$ w.r.t. just Opponent moves. Define $D(\mathcal{A}, \mathcal{B})$ to be the set of its Player V-moves.

Strategies $\mathcal{A} \twoheadrightarrow_\delta \mathcal{B}$ in SD_δ correspond to sort-respecting functions

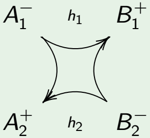
$$h : D(\mathcal{A}, \mathcal{B}) \to \mathcal{A} + \mathcal{B}$$

assigning elements of $\mathcal{A}$ and $\mathcal{B}$ to V-moves of Player. Composition à la GoI.

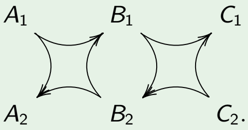
The function $h : D(\mathcal{A}, \mathcal{B}) \to \mathcal{A} + \mathcal{B}$ corresponds to a pair of stable functions

$$h_1 : A_1 \times B_2 \to A_2 \text{ and } h_2 : A_1 \times B_2 \to B_1\,,$$

from Opponent assignments A_1 and B_2 to Player assignments A_2 and B_1,



with composition



Abramsky and Jagadeesan's GoI construction w.r.t. stable domain theory.

The domains of assignments specified

Let G_V^+ be the projection of G to its Player V-moves.
Let G_V^- be the projection of G to its Opponent V-moves.

Define the domains of Player, resp. Opponent, assignments in $\mathcal{B}$ as

$$B_1 := (\mathcal{C}(\mathit{expn}(G_V^+, \mathcal{B})), \subseteq) \text{ and } B_2 := (\mathcal{C}(\mathit{expn}(G_V^-, \mathcal{B})), \subseteq).$$

E.g. the configurations of $\mathit{expn}(G_V^+, \mathcal{B})$ correspond to *assignments*, sort-respecting functions $\gamma : x \to \mathcal{B}$ from configurations $x \in \mathcal{C}(G_V^+)$.

Similarly, define the domains of assignments

$$A_1 := (\mathcal{C}(\mathit{expn}(G_V^+, \mathcal{A})), \subseteq) \text{ and } A_2 := (\mathcal{C}(\mathit{expn}(G_V^-, \mathcal{A})), \subseteq).$$

Strategies as coKleisli maps, assuming G is one-sided

$D(\mathcal{A})$ inherits Σ-structure from $\mathcal{A}$ — via the counit of δ each Player V-move e depends on an earlier corresponding assignment $\bar{e}$ of Opponent:
$R(e_1, \cdots, e_k)$ in $D(\mathcal{A})$ iff $x \models R(\bar{e}_1, \cdots, \bar{e}_k)$, some $+$-maxl config x of $D(\mathcal{A})$.
Coextension preserves homomorphisms; $D(\_)$ a comonad on Σ-structures.

When δ is copycat, the comonads $D(\_)$ and $expn(G, \_)_V$ are isomorphic.

Often, depending on the winning conditions W_G, the coKleisli category of $D(\_)$ is isomorphic to SD_δ, for example in these cases for suitable games G

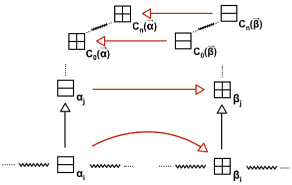
with δ as copycat, for pebbling comonads [Abramsky, Dawar, Wang]

with δ as copycat, for simulation [Abramsky, Shah]

with δ enforcing delay, for all-in-one game for trace inclusion

with δ enforcing delay, for all-in-one game of the pebble-relation comonad [Montacute, Shah]

Examples: the k-pebble game and simulation game



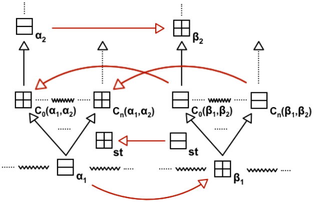


Figure: the k-pebble game (left) and the simulation game (right).

The k-pebble game $\delta_0 : \mathbb{CC}_{G_0} \to G_0^{\perp} \| G_0$ with

$$W_{G_0} \equiv \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\vec{\beta})) \to R_i(\vec{\beta}) .$$

The simulation game $\delta_1 : \mathbb{CC}_{G_1} \to G_1^{\perp} \| G_1$ with

$$W_{G_1} \equiv \mathbb{E}(st) \to Start(\beta_1) \ \wedge$$
$$\bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\beta_1, \beta_2)) \to R_i(\beta_1, \beta_2) \ \wedge \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i(\beta_2, \beta_1)) \to R_i(\beta_2, \beta_1) .$$

Example: the trace-inclusion game

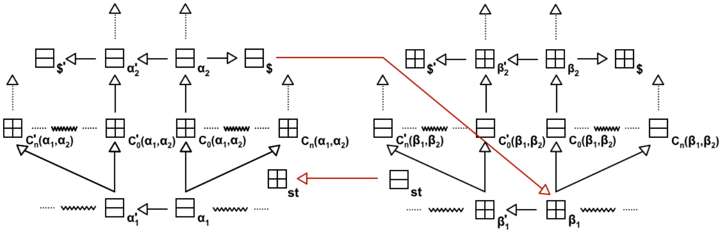


Figure: The trace-inclusion game

The trace-inclusion game $\delta_2 : D \rightarrow G_2^{\perp} \| G_2$ with

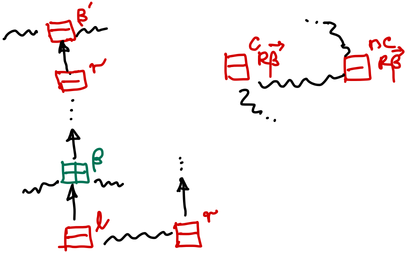
$$W_{G_2} \equiv W_{G_1} \wedge \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i'(\beta_1, \beta_2)) \rightarrow R_i'(\beta_1', \beta_2')$$

$$\wedge \bigwedge_{0 \leqslant i \leqslant n} \mathbb{E}(C_i'(\beta_2, \beta_1)) \rightarrow R_i'(\beta_2', \beta_1')$$

$$\wedge (\mathbb{E}(\beta_1') \rightarrow \beta_1 \eqslantless \beta_1') \wedge (\mathbb{E}(\beta_2') \rightarrow \beta_2 \eqslantless \beta_2') \wedge \mathbb{E}(\$) \wedge \mathbb{E}(\$').$$

Example: Ehrenfeucht-Fraïssé games

In Ehrenfeucht-Fraïssé games, $\delta_3 : \mathbb{C}_{G_3} \to G_3^{\perp} \| G_3$
where G_3 is:



— with l and r moves Opponent chooses to play in the left or right structure, with winning condition

$$W_{G_3} \equiv \Big(\bigwedge_{R\vec{\beta}} \mathbb{E}(c_{R\vec{\beta}}) \to R(\vec{\beta})\Big) \wedge \Big(\bigwedge_{R\vec{\beta}} \mathbb{E}(nc_{R\vec{\beta}}) \to \neg R(\vec{\beta})\Big)\,.$$

All-in-one variations where Opponent make all their moves before Player.