Central Submonads and Notions of Computation

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18 September 2023

What should be the role of semantics?

Recently, Samson posed interesting questions regarding the vision of mathematical semantics.

Question

"That is, should mathematical semantics still be conceived as following in the track of preexisting languages, trying to explain their novel features, and to provide firm foundations for them? Or should it be seen as operating in a more autonomous fashion, developing new semantic paradigms, which may then give rise to new languages?"

Samson Abramsky: *Whither semantics?* **Theoretical Computer Science (2020)**

Languages $\leftrightarrow$ Semantics

- Often: language first, then mathematical semantics follows.
- However, we can extract languages, abstractions and design from mathematical models:
  - Coherence spaces
    $\Rightarrow$ Linear Logic
    $\Rightarrow$ substructural types in programming (e.g. Idris 2, Haskell).
  - Monads (category theory)
    $\Rightarrow$ Moggi's monadic metalanguage
    $\Rightarrow$ monads in programming (e.g. Idris, Haskell).

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    $\Rightarrow$ monads in programming (e.g. Idris, Haskell).
- **This talk:** some recent results on monads following in this spirit.
  - Ask a question about (categorical) monads.
  - Extract design and language features from the answer.
  - How useful is it?

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Question

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What is the centre of a monad?

Some intuition

```
do                  do
  x <- op1            y <- op2
  y <- op2            x <- op1
  f x y               f x y
```

Two examples of monadic sequencing in Haskell.

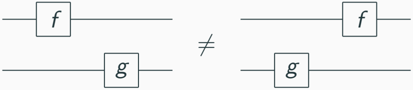
- Monads (programming) used for handling computational effects.
- Monads have an algebraic flavour (category theory).
- Centre (algebraically): elements that commute with all others.
- **Intuition:** If op1 **or** op2 is central, the two code fragments should be equivalent.

Question

What is centrality for monads?

Background: Premonoidal categories

Premonoidal categories as model of effects. Tensor $\otimes$ is not a bifunctor.



- A premonoidal category $\mathcal{P}$ has a centre $Z(\mathcal{P})$;
- $\otimes$ is a bifunctor in $Z(\mathcal{P})$;
- $Z(\mathcal{P})$ is a monoidal category.

Link with Monads

The Kleisli category of a strong monad is a premonoidal category.

J. Power & E. Robinson. *Premonoidal Categories and Notions of Computation*. **Math. Struct. Comput. Sci. (1997)**

Strong monads

Correspondence:

- Effects $\leftrightarrow$ monads;
- pairing $\leftrightarrow$ monoidal structure $\otimes$.

Strong monad: combines the two with a strength τ.

$$\tau_{X,Y} \colon X \otimes \mathcal{T} Y \to \mathcal{T}(X \otimes Y).$$

Operationally:

$$\langle x, M \rangle$$

$$\texttt{do } y \leftarrow M; \ \texttt{return } \langle x, y \rangle$$

Examples on Set

Writer monad

- Monoid M with centre $Z(M)$.
- Monad – $(M \times -) : \mathbf{Set} \to \mathbf{Set}$.
- Centre – $(Z(M) \times -)\colon \mathbf{Set} \to \mathbf{Set}$.

Powerset monad

- Commutative.
- Centre – itself.

Link with Lawvere theories

- Lawvere theory $\mathbf{T}$ with centre $Z(\mathbf{T})$.
- Monad – induced by $\mathbf{T}$.
- Centre – induced by $Z(\mathbf{T})$.

Central cone

Commutative monad $\mathcal{T}$:

$$
\begin{tikzcd}
& \mathcal{T}X\otimes\mathcal{T}Y \arrow[rr, "\tau'"] \arrow[dl, equal] & & \mathcal{T}(X\otimes\mathcal{T}Y) \arrow[d, "\mathcal{T}\tau"] \\
\mathcal{T}X\otimes\mathcal{T}Y \arrow[d, "\tau"'] & & & \mathcal{T}^2(X\otimes Y) \arrow[d, "\mu"] \\
\mathcal{T}(\mathcal{T}X\otimes Y) \arrow[r, "\mathcal{T}\tau'"'] & \mathcal{T}^2(X\otimes Y) \arrow[rr, "\mu"'] & & \mathcal{T}(X\otimes Y)
\end{tikzcd}
$$

commutes for all objects X and Y.

*Central cone of $\mathcal{T}$ at **fixed** X:*
a pair $(Z, \iota\colon Z \to \mathcal{T}X)$ such that

$$
\begin{tikzcd}
Z\otimes\mathcal{T}Y \arrow[r, "\iota\otimes\mathcal{T}Y"] \arrow[d, "\iota\otimes\mathcal{T}Y"'] & \mathcal{T}X\otimes\mathcal{T}Y \arrow[r, "\tau'"] & \mathcal{T}(X\otimes\mathcal{T}Y) \arrow[d, "\mathcal{T}\tau"] \\
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If the universal central cone at X exists, write $\mathcal{Z}X \stackrel{\text{def}}{=} Z$.

Central cone

Commutative monad $\mathcal{T}$:

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\begin{tikzcd}
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Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad $\mathcal{T}$ to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a commutative submonad $\mathcal{Z}$, s.t. $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \to \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \to Z(\mathbf{C}_{\mathcal{T}})$.

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Corollaries

(1) All strong monads on $\mathbf{Set}, \mathbf{DCPO}, \mathbf{Top}, \mathbf{Vect}, \ldots$ are centralisable.

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All strong monads centralisable? No, but only artificial counterexamples!

Even more categories with centralisable monads

All strong monads on the following categories are centralisable:

- EM-category of a commutative monad on **Set**. This includes:
  - **Conv** of convex sets (algebras of the distribution monad),
  - $\mathbf{Set}_*$ - pointed sets and point preserving functions (lift monad),
  - **CMon** - commutative monoids (commutative monoid monad),
  - **Sup** - complete semilattices (algebras of the powerset monad).
- Presheaf category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ over a small category $\mathbf{C}$. This includes:
  - **Graph** - directed multi-graphs and graph homomorphisms,
  - $\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ - topos of trees,
  - $\mathbf{G} - \mathbf{Set}$ - G-sets (sets with an action of G) and equivariant maps.
- Any Grothendieck topos.
- And probably many more.

Another (non-trivial) centre

Example

Every semiring $(S, +, 0, \cdot, 1)$ induces a monad $\mathcal{T}_S : \mathbf{Set} \to \mathbf{Set}$ [Jakl et al., 2022]. This monad maps a set X to the set of finite formal sums of the form $\sum s_i x_i$, where s_i are elements of S and x_i are elements of X. The centre $\mathcal{Z}$ of $\mathcal{T}_S$ is induced by the commutative semiring $Z(S)$, i.e., by the centre of S in the usual sense.

A (trivial?) domain-theoretic example

Example

The valuation monad $\mathcal{V}\colon \mathbf{DCPO} \to \mathbf{DCPO}$ is strong, but its commutativity is an open problem. The centre of $\mathcal{V}$ is precisely the "central valuations monad". A central cone at X is determined by:

$$\mathcal{Z}X \stackrel{\text{def}}{=} \Big\{ \xi \in \mathcal{V}(X) \mid \forall Y \in \mathrm{Ob}(\mathbf{DCPO}). \forall U \in \sigma(X \times Y).$$

$$\forall \nu \in \mathcal{V}(Y). \int_X \int_Y \chi_U(x,y) d\nu d\xi = \int_Y \int_X \chi_U(x,y) d\xi d\nu \Big\}$$

together with the subset inclusion into $\mathcal{V}X$.

C. Jones & G. D. Plotkin. *A probabilistic powerdomain of evaluations.* **LICS 1989.**
X. Jia, M. W. Mislove & V. Zamdzhiev. *The Central Valuations Monad (Early Ideas).* **CALCO 2021**

Central Submonad

Theorem

Given a submonad $\mathcal{S}$ with $\iota\colon \mathcal{S} \hookrightarrow \mathcal{T}$, equivalent conditions:

1. (S_X, ι_X) is a central cone for all X;
2. there exists a canonical embedding $\mathbf{C}_{\mathcal{S}} \hookrightarrow Z(\mathbf{C}_{\mathcal{T}})$.

When the centre $\mathcal{Z}$ exists, also equivalent to:

3. $\mathcal{S}$ is (canonically) a submonad of $\mathcal{Z}$.

Definition: $\mathcal{S}$ – *central* submonad if it satisfies 1, 2 or 3.

Remark: the centre is the universal central submonad.

Computational Interpretation

Language	Model
Simply-typed λ-calculus (STλC)	Cartesian Closed Category (CCC)
Moggi's metalanguage	CCC with strong monad $\mathcal{T}$
???	CCC with central submonad $\mathcal{S} \hookrightarrow \mathcal{T}$

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Simply-typed λ-calculus (STλC)	Cartesian Closed Category (CCC)
Moggi's metalanguage	CCC with strong monad $\mathcal{T}$
CSC	CCC with central submonad $\mathcal{S} \hookrightarrow \mathcal{T}$

CSC (Central Submonad Calculus):

Simply-typed λ-calculus;

$+$ new types: $\mathcal{S}X$ and $\mathcal{T}X$;

$+$ terms for monadic computation (à la Moggi);

$+$ equational rules, such as:

$$\begin{aligned}\Gamma \vdash\ & \mathtt{do}_{\mathcal{T}}\ x \leftarrow \iota M;\ \mathtt{do}_{\mathcal{T}}\ y \leftarrow N;\ P \\ &= \mathtt{do}_{\mathcal{T}}\ y \leftarrow N;\ \mathtt{do}_{\mathcal{T}}\ x \leftarrow \iota M;\ P : \mathcal{T}C\end{aligned}$$

Completeness and Internal Language

Folklore:

$$\begin{tikzcd}[column sep=huge] \mathsf{ST\lambda Cs} \arrow[rr, bend left, "\text{Completeness}"] \arrow[r, phantom, "\wr"] & {} & \mathsf{CCCs} \arrow[ll, bend left, "\text{Internal Language}"] \end{tikzcd}$$

Theorem:

$$\begin{tikzcd}[column sep=huge] \mathsf{CSCs} \arrow[rr, bend left, "\text{Completeness}"] \arrow[r, phantom, "\wr"] & {} & \mathsf{CSC\text{-}models} \arrow[ll, bend left, "\text{Internal Language}"] \end{tikzcd}$$

Conclusion

What we have done:

- notion of centre for strong monads;
- equivalent conditions for a strong monad to have a centre;
- equivalent conditions for a submonad to be central;
- computational interpretation: completeness and internal language.

Further questions:

- more interesting examples?
- how to use in practice (e.g. program logic)?

More details:

- Paper at LICS'23 and `https://arxiv.org/abs/2207.09190`,
- the PhD thesis of Louis, available before September 2024.

Jakl, T., Marsden, D., and Shah, N. (2022).
Generalizations of bilinear maps – technical report.